LOAD FLOW ANALYSIS

1 Problem definition

In electrical power networks, several planning problems must be studied and solved for satisfactory, reliable, and high-quality operation. One of the important problems studied—both in planning (for potential network expansions) and during operation—is power flow or load flow calculation.

Power flow calculation involves determining all power transits and voltages in the network for a given load case. It is defined by a set of equations derived from Kirchhoff's laws and applied to the network buses. Additionally, operational constraints on the network are taken into account. The problem is nonlinear because power is the product of voltage and current. A steady-state sinusoidal regime is assumed (thus algebraic rather than differential equations), with operating conditions fixed for each study.

2 Power flow equations

Consider the representation in Fig. 1 below for an n-bus electrical network.

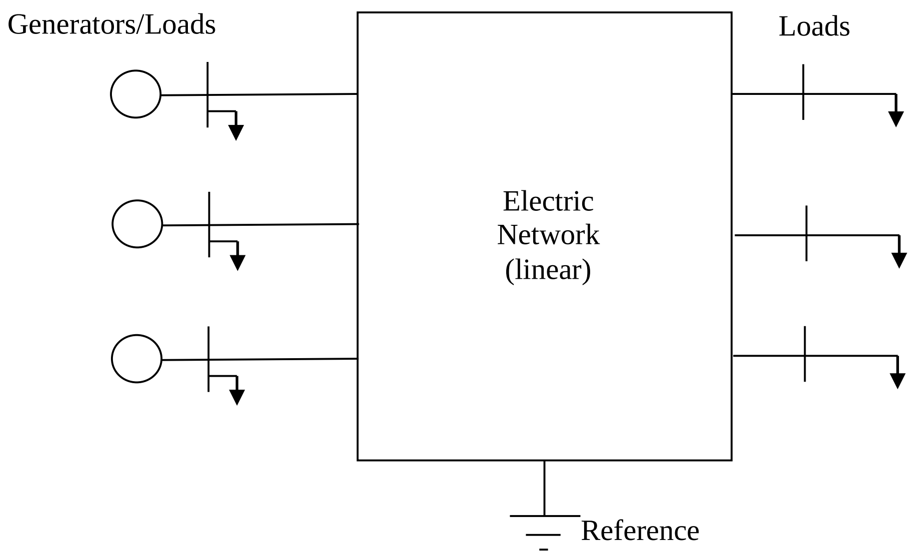

Fig. 1: Schematic representation of an electrical network

A node or bus (busbar) is a point in the electrical network where apparent power is either consumed by a load or delivered by a generator connected to that bus. Buses connect the lines and thus form the sources and loads of the electrical network.

Indeed, in addition to lines and transformers, the possible elements connected to a bus are generators, loads, and shunt elements.

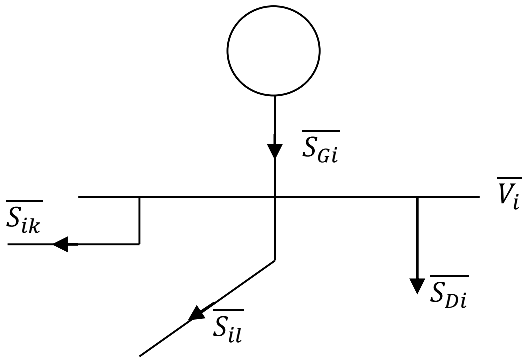

Fig. 2 : A bus i of the network connected to other buses k et l

For this bus, we can define the injected power:

$$\overline{S_i} = \overline{S_{Gi}} - \overline{S_{Di}} \quad (1)$$

where S_{Gi} is the power generated at bus i and S_{Di} is the power consumed or requested at bus i.

The law of conservation of power states:

$$\overline{S_i} = \sum_{j=1}^{n} \overline{S_{ij}} \qquad i = 1,2,\dots,n \quad (2)$$

where S_{ij} is the power that flows in the line ij connecting buses i and j.

The current injected at bus i (Fig. 3):

$$I_i = I_{Gi} - I_{Di} = \sum_{j=0}^{n} I_{ij} \qquad i = 1,2,\dots,n \quad (3)$$

with:

$$I_{ij} = y_{ij}(V_i - V_j) \quad (4)$$

and $V_j = 0$ for $j = 0$ (reference bus).

Hence:

$$I_i = \sum_{j=0}^{n} y_{ij}(V_i - V_j) = V_i \sum_{j=0}^{n} y_{ij} - \sum_{j=1, i\neq j}^{n} y_{ij}V_j \quad (5)$$

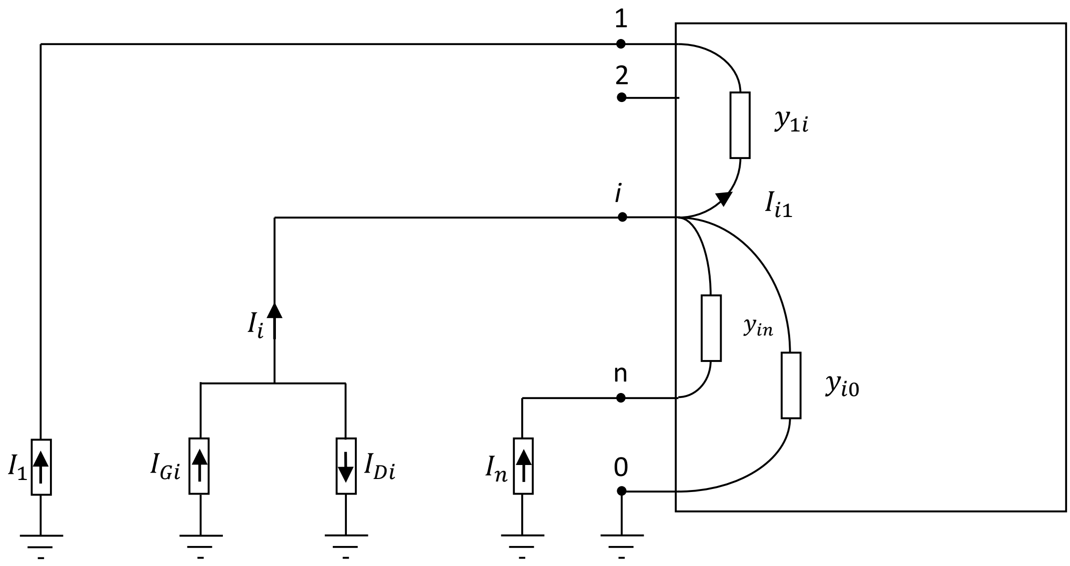

Fig. 3: Representation of the electrical network with injected currents and admittances

In matrix form:

$$\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_i \\ \vdots \\ I_n \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^{n} y_{1j} & -y_{12} & \cdots & \cdots & -y_{1n} \\ -y_{21} & \sum_{j=0}^{n} y_{2j} & \cdots & \cdots & -y_{2n} \\ & & \vdots & & \\ -y_{i1} & -y_{i2} & \cdots \sum_{j=0}^{n} y_{ij} & \cdots & -y_{in} \\ & & \vdots & & \\ -y_{n1} & -y_{n2} & \cdots & \cdots & \sum_{j=0}^{n} y_{nj} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_i \\ \vdots \\ V_n \end{bmatrix} \quad (6)$$

In compact form:

$$[I] = [Y_{bus}][V] \quad (7)$$

Y_{bus} is the nodal admittance matrix. For simplicity, it will be denoted simply as Y. Y_{bus} is also called the short-circuit admittance matrix. Its elements are calculated by:

$$\begin{cases} Y_{ij} = -y_{ij} & i \neq j \\ Y_{ii} = \sum_{j=0}^{n} y_{ij} \end{cases} \quad (8)$$

The complex power injected at bus i is written as:

$$\overline{S_i} = \overline{V_i}\,\overline{I_i}^* = \overline{V_i}\left(\sum_{j=1}^{n} Y_{ij}\overline{V_j}\right)^* = \overline{V_i}\sum_{j=1}^{n} Y_{ij}^*\overline{V_j}^* \quad i = 1,2,\dots,n \tag{9}$$

Several formulations are possible because the complex elements of Y, or complex voltages and currents, can be written in rectangular or polar coordinates.

By writing the voltages in the form:

$$\bar{V}_i = V_i e^{j\delta_i} \quad \text{and} \quad \bar{V}_j = V_j e^{j\delta_j}$$

By expressing the element (i, j) of the matrix Y by

$$Y_{ij} = G_{ij} + jB_{ij}$$

and noting: $\delta_{ij} = \delta_i - \delta_j$

We obtain:

$$\begin{aligned}\overline{S_i} = P_i + jQ_i &= \textstyle\sum_{j=1}^{n} V_i V_j e^{j\delta_{ij}}\left(G_{ij} - jB_{ij}\right)\\ &= \sum_{j=1}^{n} V_i V_j\left(cos\delta_{ij} + jsin\delta_{ij}\right)\left(G_{ij} - jB_{ij}\right) \quad i = 1,2,\dots,n\end{aligned} \tag{10}$$

The active and reactive power injected at the buses can then be written as:

$$\begin{cases} P_i = V_i \sum_{j=1}^{n} V_j\left(G_{ij}cos\delta_{ij} + B_{ij}sin\delta_{ij}\right) \\ Q_i = V_i \sum_{j=1}^{n} V_j\left(G_{ij}sin\delta_{ij} - B_{ij}cos\delta_{ij}\right) \end{cases} \quad i = 1,2,\dots,n \tag{11}$$

These are the power flow equations where the injected powers are known, and the magnitudes and angles of the voltages at the various buses must be determined.
The resulting power flow equations are nonlinear and cannot be solved analytically. Iterative methods are used: this is a repetitive operation that starts with an initial solution estimate, which is combined with the original equations to compute a first solution, then a second, and so on until a satisfactory solution is reached.

In fact, the power flow problem may have no solution, one solution, or multiple solutions. Four quantities are associated with each bus in the system: active and reactive powers, as well as the magnitude and phase of the voltage. Only two of these four variables are known at a bus, with the other two determined during the calculation. Three combinations, defining three types of buses, are generally used:

- Buses PQ for which active and reactive power injections are fixed: A bus PQ is directly connected to a load and has no energy source.

- Buses PV where the known quantities are active power and voltage magnitude: A PV bus is directly connected to a generator or reactive energy source. Reactive energy production is limited by lower and upper values, Q_{Gmin} et Q_{Gmax}, respectively. If one of these limits is reached, the value is fixed at that limit, and the voltage is released; the bus then becomes a PQ bus.

- The swing or slack bus, for which the complex voltage is given in magnitude and phase, serving as the system reference: This bus is included in the power flow calculation to compensate for active losses and ensure equality between active power demand and generation. Without defining this bus, convergence—and thus a solution—is impossible.

3 Problem-solving methods

3.1 Gauss-Seidel method

The Gauss-Seidel method is an iterative method used to solve systems of linear equations with multiple unknowns. This method first assigns initial values to the unknowns for the first calculation of results. These new values are then used as initial data for a second iteration, and so on. The iterative process stops once the difference between values obtained from two consecutive iterations is less than a tolerance denoted by ε.

3.1.1 Principle of the method

Consider the system of equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ \vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n \end{cases} \tag{12}$$

This system of equations can be written in another form where we derive the expression for x_i from each equation:

$$\begin{cases} x_1 = \dfrac{b_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n}{a_{11}} \\ x_2 = \dfrac{b_2 - a_{21}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n}{a_{22}} \\ \vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \\ x_n = \dfrac{b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{nn-1}x_{n-1}}{a_{nn}} \end{cases} \tag{13}$$

$$a_{ii} \neq 0, \qquad i = 1,2,\cdots,n$$

If we denote by k the number of iterations, this method consists of evaluating x^{k+1} from x^k. The previous system of equations can then be written.

$$\begin{cases} {x_1}^{k+1} = \dfrac{b_1 - a_{12}{x_2}^k - a_{13}{x_3}^k - \cdots - a_{1n}{x_n}^k}{a_{11}} \\ {x_2}^{k+1} = \dfrac{b_2 - a_{21}{x_1}^k - a_{23}{x_3}^k - \cdots - a_{2n}{x_n}^k}{a_{22}} \\ \vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \\ {x_n}^{k+1} = \dfrac{b_n - a_{n1}{x_1}^k - a_{n2}{x_2}^k - \cdots - a_{nn-1}{x_{n-1}}^k}{a_{nn}} \end{cases} \tag{14}$$

which can also be written:

$$ {x_i}^{k+1} = \frac{b_i - a_{i1}{x_1}^k - a_{i2}{x_2}^k - \cdots - a_{in}{x_n}^k}{a_{ii}} \quad i = 1,2,\cdots,n \tag{15}$$

or:

$$x_i^{k+1} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1, j\neq i}^{n} a_{ij} x_j^k\right) \qquad i = 1,2,\cdots,n \tag{16}$$

The Gaussian method, also known as the Jacobi method, requires a vector of initial values to begin the iterations.

$$x^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{bmatrix}$$

The iteration process will stop once the convergence condition is met:

$$\left|x_i^{k+1} - x_i^k\right| \leq \varepsilon \qquad i = 1,2,\cdots,n \tag{17}$$

The Gauss-Seidel method is an improvement on the Jacobi method which consists of calculating the unknown at iteration $k+1$, x^{k+1} as a function of the values x^k of iteration k and x^{k+1} available at iteration $k+1$ as follows:

$$x_i^{k+1} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{k+1} - \sum_{j=i+1}^{n} a_{ij} x_j^k\right) \qquad i = 1,2,\cdots,n \tag{18}$$

The number of iterations of the Gauss-Seidel method can be improved by using an acceleration factor α at each iteration as follows:

$$x_i^{k+1} = x_i^k + \alpha\left(x_i^{k+1} - x_i^k\right) \tag{19}$$

3.1.2 Application to the power flow problem

The power flow calculation equation leads us back to the calculation of voltages at the network buses.

$$\bar{I_i} = \sum_{j=1}^{n} Y_{ij} \overline{V_j} \tag{20}$$

$$\overline{S_i} = P_i + jQ_i = \overline{V_i}\, \overline{I_i}^* = \overline{V_i}\left(\sum_{j=1}^{n} Y_{ij} \overline{V_j}\right)^* \tag{21}$$

Taking the conjugate of the apparent power, we have:

$$\overline{S_i}^* = P_i - jQ_i = \overline{V_i}^* \cdot \overline{I_i} = \overline{V_i}^* \cdot Y_{ii} \cdot \overline{V_i} + \overline{V_i}^*\left(\sum_{j=1}^{i-1} Y_{ij} \overline{V_j}\right) + \overline{V_i}^*\left(\sum_{j=i+1}^{n} Y_{ij} \overline{V_j}\right) \tag{22}$$

The Gauss-Seidel iteration process can then be applied as follows:

$$\overline{V_i}^{k+1} = \frac{1}{Y_{ii}}\left(\frac{\overline{S_i}^*}{\overline{V_i}^{*k}} - \sum_{j=1}^{i-1} Y_{ij} \overline{V_j}^{k+1} - \sum_{j=i+1}^{n} Y_{ij} \overline{V_j}^{k}\right) \tag{23}$$

This equation does not apply to the swing bus where V and δ are given; P and Q are to be determined.

For PV buses, the reactive power is unknown. It must be calculated using the following equation:

$$Q_i^{k+1} = -imag\left\{\overline{V_i}^{*k} \cdot \left(Y_{ii} \cdot \overline{V_i}^k + \sum_{j=1}^{i-1} Y_{ij}\overline{V_j}^{k+1} + \sum_{j=i+1}^{n} Y_{ij}\overline{V_j}^k\right)\right\} \tag{24}$$

If the value calculated from the previous equation violates one of the specified limits, the value of Q_i is set to the violated limit. Then, the calculation of $\overline{V_i}^{k+1}$ is performed as before for a PQ bus.

For the calculation of $\overline{V_i}^{k+1}$ for this type of bus, since the voltage magnitude is specified, only the phase angle δ calculated by performing the following operation needs to be retained:

$$\overline{V_i}^{k+1} = \frac{\overline{V_i}^{k+1}}{\left|\overline{V_i}^{k+1}\right|} \cdot \left|\bar{V}_{i,specified}\right| \tag{25}$$

To accelerate the convergence of the method, the voltages during successive iterations must be modified as follows:

$$\bar{V}_i^{k+1} = \bar{V}_i^k + \alpha\left(\bar{V}_i^{k+1} - \bar{V}_i^k\right) \tag{26}$$

where α is the acceleration factor.
For most electrical networks, α is such that: $1{,}1 \leq \alpha \leq 2$

3.2 Newton-Raphson method

Among the various existing methods for solving a load flow calculation, the Newton-Raphson method is the one that is commonly and widely used.

3.2.1 Principle of the method

Consider the system:

$$f(x) = 0 \tag{27}$$

with:

$$x = [x_1\ x_2\ \dots\ x_n]^t$$

$$f = [f_1\ f_2\ \dots\ f_n]^t$$

The iterative Newton-Raphson solution is given by:

$$x^{k+1} = x^k - \left[J\left(x^k\right)\right]^{-1}. f(x^k) \tag{28}$$

where:

x^k : solution at iteration k
x^{k+1} : solution at iteration $k+1$
J : system Jacobian

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots\dots\dots\dots\dots\dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots\dots\dots\dots\dots\dots & \frac{\partial f_2}{\partial x_n} \\ & & \vdots & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots\dots\dots\dots\dots\dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \quad (29)$$

We often write the resolution in the form:

$$x^{k+1} = x^k + \Delta x^k \quad (30)$$

$$\Delta x^k = -\left[J\left(x^k\right)\right]^{-1} . f(x^k) \quad (31)$$

3.2.2 Application to the power flow problem

We have the power flow equations which take the form:

$$\Delta P_i = P_{ispec} - P_i \quad (32)$$

$$\Delta Q_i = Q_{ispec} - Q_i \quad (33)$$

with $i = 1,2, \dots, n$ and bus 1 considered as the swing bus.
P_{ispec} et Q_{ispec} are the injected powers specified in each bus (generation and load).

The system to be solved is written as:

$$f(x) = \begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix} = 0 \quad (34)$$

with:

$$\Delta P = [\Delta P_2 \;\; \Delta P_3 \;\; \dots \;\; \Delta P_n]^t$$

$$\Delta Q = [\Delta Q_2 \;\; \Delta Q_3 \;\; \dots \;\; \Delta Q_n]^t$$

$$x = [\delta \;\; V]^t = [\delta_2 \;\; \delta_3 \dots \delta_n \;\; V_2 \;\; V_3 \dots V_n]^t$$

The linear system to solve to find the increments:

$$\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} \Delta\delta \\ \Delta V \end{bmatrix} = \begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix} \quad (35)$$

with elements of the Jacobian:

$$J_{11} = -\frac{\partial \Delta P}{\partial \delta} = \frac{\partial P}{\partial \delta} \quad (36)$$

$$J_{12} = -\frac{\partial \Delta P}{\partial V} = \frac{\partial P}{\partial V} \quad (37)$$

$$J_{21} = -\frac{\partial \Delta Q}{\partial \delta} = \frac{\partial Q}{\partial \delta} \quad (38)$$

$$J_{22} = -\frac{\partial \Delta Q}{\partial V} = \frac{\partial Q}{\partial V} \quad (39)$$

Using the expressions from the power flow equations:

$$\Delta P_i = P_{ispec} - \sum_{j=1}^{n} V_i V_j \big(G_{ij}\cos(\delta_i - \delta_j) + B_{ij} sin(\delta_i - \delta_j)\big) \quad (40)$$

$$\Delta Q_i = Q_{ispec} - \sum_{j=1}^{n} V_i V_j \big(G_{ij} sin(\delta_i - \delta_j) - B_{ij} cos(\delta_i - \delta_j)\big) \quad (41)$$

We can calculate the expressions for the different elements of the Jacobian:

$$(J_{11})_{ij} = -\frac{\partial \Delta P_i}{\partial \delta_j} = V_i V_j \big(G_{ij} sin\delta_{ij} - B_{ij} cos\delta_{ij}\big) \quad (42)$$

$$(J_{11})_{ii} = -\frac{\partial \Delta P_i}{\partial \delta_i} = \sum_{j=1, j\neq i}^{n} V_i V_j \big(-G_{ij} sin\delta_{ij} + B_{ij} cos\delta_{ij}\big) = -Q_i - B_{ii} V_i^2 \quad (43)$$

$$(J_{12})_{ij} = -\frac{\partial \Delta P_i}{\partial V_j} = V_i \big(G_{ij} cos\delta_{ij} + B_{ij} sin\delta_{ij}\big) \quad (44)$$

$$(J_{12})_{ii} = -\frac{\partial \Delta P_i}{\partial V_i} = \sum_{j=1, j\neq i}^{n} V_j \big(G_{ij} cos\delta_{ij} + B_{ij} sin\delta_{ij}\big) + 2V_i G_{ii} = \frac{P_i}{V_i} + G_{ii} V_i \quad (45)$$

$$(J_{21})_{ij} = -\frac{\partial \Delta Q_i}{\partial \delta_j} = V_i V_j \big(-G_{ij} cos\delta_{ij} - B_{ij} sin\delta_{ij}\big) = -V_j (J_{12})_{ij} \quad (46)$$

$$(J_{21})_{ii} = -\frac{\partial \Delta Q_i}{\partial \delta_i} = \sum_{j=1, j\neq i}^{n} V_i V_j \big(G_{ij} cos\delta_{ij} + B_{ij} sin\delta_{ij}\big) = P_i - G_{ii} V_i^2 \quad (47)$$

$$(J_{22})_{ij} = -\frac{\partial \Delta Q_i}{\partial V_j} = V_i \big(G_{ij} sin\delta_{ij} - B_{ij} cos\delta_{ij}\big) = \frac{(J_{11})_{ij}}{V_j} \quad (48)$$

$$(J_{22})_{ii} = -\frac{\partial \Delta Q_i}{\partial V_i} = \sum_{j=1, j\neq i}^{n} V_j \big(G_{ij} sin\delta_{ij} - B_{ij} cos\delta_{ij}\big) - 2V_i B_{ii} = \frac{Q_i}{V_i} - B_{ii} V_i \quad (49)$$

3.2.3 Calculation algorithm

It can be summarized in the following steps:

1. Read the system data and form the admittance matrix Y.
2. Iteration: Estimate an initial solution. $x^0 = [\delta^0 \;\; V^0]$
   A typical initial solution is the "flat start": $V_i^0 = 1$; $\delta_i^0 = 0\, pu$ for all buses i.
3. Evaluate the "mismatches" ΔP et ΔQ for the current solution.
4. Check the calculation tolerance (precision or stopping criterion):
   $|\Delta P_i| \leq \varepsilon$ and $|\Delta Q_i| \leq \varepsilon$ $\; i = 2,3, \dots, n$ or $max(|\Delta P|, |\Delta Q|) \leq \varepsilon$ where ε is a chosen tolerance
   If yes: stop (end of iterations).
   If no: go to step 5.
5. Calculate the Jacobian elements for the current solution.

6. Solve the linear system (35) for $\Delta x = [\Delta\delta \ \ \Delta V]$.
7. Update the solution: $x^{k+1} = x^k + \Delta x^k$
   $\delta^{k+1} = \delta^k + \Delta\delta^k$ and $V^{k+1} = V^k + \Delta V^k$
8. Increment iteration and return to step 3.

3.2.4 Voltage-controlled buses

If for a bus i the voltage magnitude is controlled and thus kept constant ($V_i = const$) during the power flow calculation, then the column in the Jacobian corresponding to the derivatives with respect to V_i is eliminated. This is the case for PV buses where the voltage magnitude is specified. At these buses, the injected reactive power Q will be calculated after solution and convergence.

If we have m PV buses, then there will be $(n-1-m)$ load or PQ buses. The dimension of the Jacobian is then $(n-1-m) \times (n-1-m)$ instead of $(n-1) \times (n-1)$ relative to the case where all buses except the slack bus are of PQ type.

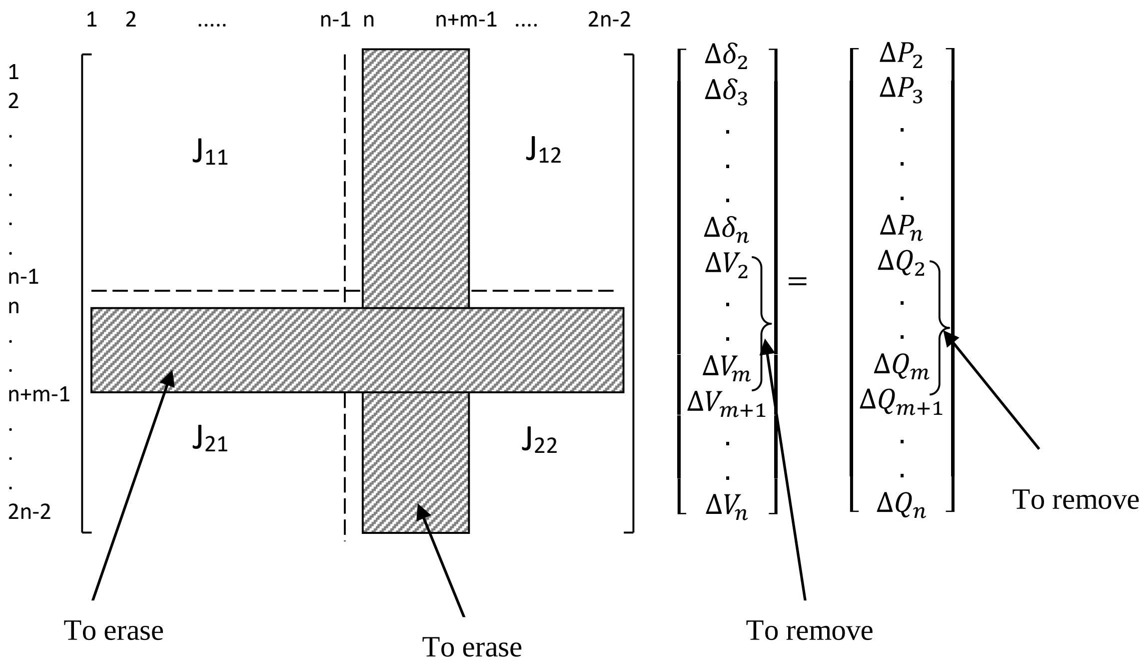

Fig. 4 : Modification of the Jacobian matrix taking into account PV buses

Notes:

- If during the calculation, a generator reaches its reactive power production limit, it is converted to a load bus and is no longer voltage-controlled. For this, the injected reactive power at PV buses is determined at each iteration, and it is verified that it remains within the specified limits.
- After convergence and obtaining the solution, it is possible to calculate the active and reactive powers generated at the slack bus, the reactive powers produced at PV buses, the power flowing in lines and through transformers, as well as power losses and voltage drops.

- The number of Newton-Raphson method iterations is independent of the number of buses in the network to be calculated, but computing the Jacobian consumes a lot of calculation time. The number of iterations is reduced due to the method's quadratic convergence. Overall, the Newton-Raphson method is advantageous compared to other methods.

3.3 Example

Solve the power flow problem of the 3-bus system below (Fig. 5) using the Newton-Raphson method. The impedances of all lines in pu per km are equal to $(31{,}1 + j316).10^{-6}\ pu/km$.

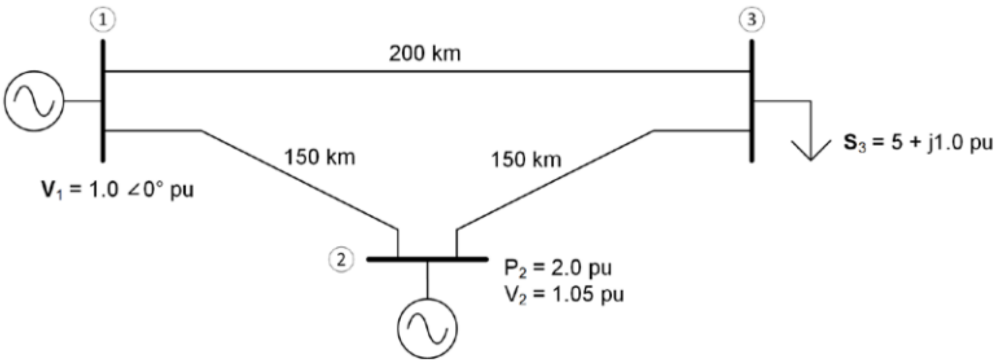

Fig. 5: 3-bus electrical network

Classification of buses:

- Bus 1 : swing (slack bus), V_1 and δ_1 known, find P_1 and Q_1
- Bus 2 : PV, P_2 and V_2 known, find δ_2 and Q_2
- Bus 3 : PQ, P_3 and Q_3 known, find V_3 and δ_3

Line impedances:

$$z_{12} = (31{,}1 + j316).10^{-6} \times 150 = (4{,}665 + j47{,}4).10^{-3}\ pu$$
$$z_{13} = (31{,}1 + j316).10^{-6} \times 200 = (6{,}220 + j63{,}2).10^{-3}\ pu$$
$$z_{23} = (31{,}1 + j316).10^{-6} \times 150 = (4{,}665 + j47{,}4).10^{-3}\ pu$$

Line admittances:

$$y_{12} = y_{23} = \frac{1}{z_{12}} = \frac{1}{(4{,}665 + j47{,}4).10^{-3}} = 2{,}06 - j20{,}9\ pu$$
$$y_{13} = \frac{1}{z_{13}} = \frac{1}{(6{,}220 + j63{,}2).10^{-3}} = 1{,}54 - j15{,}7\ pu$$

The admittance matrix:

$$Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix} = \begin{bmatrix} 3{,}6 - j36{,}6 & -2{,}06 + j20{,}9 & -1{,}5 + j15{,}7 \\ -2{,}06 + j20{,}9 & 4{,}1 - j41{,}8 & -2{,}06 + j20{,}9 \\ -1{,}5 + j15{,}7 & -2{,}06 + j20{,}9 & 3{,}6 - j36{,}6 \end{bmatrix}$$

Either:

$$G = \begin{bmatrix} 3{,}6 & -2{,}06 & -1{,}5 \\ -2{,}06 & 4{,}1 & -2{,}06 \\ -1{,}5 & -2{,}06 & 3{,}6 \end{bmatrix} \qquad B = \begin{bmatrix} -36{,}6 & 20{,}9 & 15{,}7 \\ 20{,}9 & -41{,}8 & 20{,}9 \\ 15{,}7 & 20{,}9 & -36{,}6 \end{bmatrix}$$

Specified quantities:

- $Slack$ bus : $V_1 = 1{,}0\ pu,\ \ \delta_1 = 0°$
- PV bus : $V_2 = 1{,}05\ pu, P_2 = 2{,}0\ pu$
- PQ bus : $P_3 = -5{,}0\ pu, Q_3 = -1{,}0\ pu$

Output vector (power mismatches):

$$\begin{bmatrix}\Delta \boldsymbol{P}\\ \Delta \boldsymbol{Q}\end{bmatrix} = \begin{bmatrix}\Delta P_2\\ \Delta P_3\\ \Delta Q_3\end{bmatrix}$$

$$\Delta P_2 = 2{,}0 - [(V_2V_1(G_{21}\cos(\delta_2-\delta_1) + B_{21}\sin(\delta_2-\delta_1)) + V_2^2G_{22} + V_2V_3(G_{23}\cos(\delta_2-\delta_3) + B_{23}sin(\delta_2-\delta_3))]$$

$$\Delta P_3 = -5{,}0 - [(V_3V_1(G_{31}\cos(\delta_3-\delta_1) + B_{31}\sin(\delta_3-\delta_1)) + V_3V_2(G_{32}\cos(\delta_3-\delta_2) + B_{32}\sin(\delta_3-\delta_2)) + V_3^2G_{33}]$$

$$\Delta Q_3 = -1{,}0 - [(V_3V_1(G_{31}\sin(\delta_3-\delta_1) - B_{31}\cos(\delta_3-\delta_1)) + V_3V_2(G_{32}\sin(\delta_3-\delta_2) - B_{32}\cos(\delta_3-\delta_2)) - V_3^2B_{33}]$$

Vector of unknown quantities:

$$x = \begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{V}\end{bmatrix} = \begin{bmatrix}\delta_2\\ \delta_3\\ V_3\end{bmatrix}$$

The initial vector formed by the estimated initial values:

$$x^0 = \begin{bmatrix}{\delta_2}^0\\ {\delta_3}^0\\ {V_3}^0\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 1{,}0\end{bmatrix}$$

The complex voltage vector is then:

$$\boldsymbol{V} = \begin{bmatrix}V_1\angle\delta_1\\ V_2\angle\delta_2\\ V_3\angle\delta_3\end{bmatrix} = \begin{bmatrix}1{,}0\angle 0°\\ 1{,}05\angle\delta_2\\ V_3\angle\delta_3\end{bmatrix} = \begin{bmatrix}1{,}0\angle 0°\\ 1{,}05\angle 0°\\ 1{,}0\angle 0°\end{bmatrix}$$

The Jacobian matrix:

$$J = \begin{bmatrix}\frac{\partial P_2}{\partial \delta_2} & \frac{\partial P_2}{\partial \delta_3} & \frac{\partial P_2}{\partial V_3}\\ \frac{\partial P_3}{\partial \delta_2} & \frac{\partial P_3}{\partial \delta_3} & \frac{\partial P_3}{\partial V_3}\\ \frac{\partial Q_3}{\partial \delta_2} & \frac{\partial Q_3}{\partial \delta_3} & \frac{\partial Q_3}{\partial V_3}\end{bmatrix}$$

$$\frac{\partial P_2}{\partial \delta_2} = -V_2[V_1(-G_{21}\sin(\delta_2-\delta_1) + B_{21}\cos(\delta_2-\delta_1)) + V_3(-G_{23}\sin(\delta_2-\delta_3) + B_{23}\cos(\delta_2-\delta_3))]$$

$$\frac{\partial P_3}{\partial \delta_3} = -V_3[V_1(-G_{31}\sin(\delta_3-\delta_1) + B_{31}\cos(\delta_3-\delta_1)) + V_2(-G_{32}\sin(\delta_3-\delta_2) + B_{32}\cos(\delta_3-\delta_2))]$$

$$\frac{\partial P_2}{\partial \delta_3} = V_2V_3(G_{23}\sin(\delta_2-\delta_3) - B_{23}\cos(\delta_2-\delta_3))$$

$$\frac{\partial P_3}{\partial \delta_2} = V_2V_3(G_{32}\sin(\delta_3-\delta_2) - B_{32}\cos(\delta_3-\delta_2))$$

$$\frac{\partial P_2}{\partial V_3} = V_2(G_{23}\cos(\delta_2-\delta_3) + B_{23}\sin(\delta_2-\delta_3))$$

$$\frac{\partial P_3}{\partial V_3} = V_1(G_{31}\cos(\delta_3-\delta_1) + B_{31}\sin(\delta_3-\delta_1)) + V_2(G_{32}\cos(\delta_3-\delta_2) + B_{32}\sin(\delta_3-\delta_2)) + 2V_3G_{33}$$

$$\frac{\partial Q_3}{\partial \delta_2} = V_2V_3(-G_{32}\cos(\delta_3-\delta_2) - B_{32}\sin(\delta_3-\delta_2))$$

$$\frac{\partial Q_3}{\partial \delta_3} = V_3[V_1(G_{31}\cos(\delta_3-\delta_1) + B_{31}\sin(\delta_3-\delta_1)) + V_2(G_{32}\cos(\delta_3-\delta_2) + B_{32}\sin(\delta_3-\delta_2))]$$

$$\frac{\partial Q_3}{\partial V_3} = V_1(G_{31}\sin(\delta_3-\delta_1) - B_{31}\cos(\delta_3-\delta_1)) + V_2(G_{32}\sin(\delta_3-\delta_2) - B_{32}\cos(\delta_3-\delta_2)) - 2V_3B_{33}$$

- Parameter to be specified: tolerance or precision $\varepsilon = 10^{-6}$
- Stopping criterion : $max(|\Delta P|, |\Delta Q|) \le \varepsilon$

Iteration 0 :

$$\boldsymbol{V^0} = \begin{bmatrix} 1{,}0\angle 0° \\ 1{,}05\angle 0° \\ 1{,}0\angle 0° \end{bmatrix}$$

$$\begin{bmatrix} \Delta \boldsymbol{P} \\ \Delta \boldsymbol{Q} \end{bmatrix} = \begin{bmatrix} 2{,}0 - 0{,}216 \\ -5{,}0 - (-0{,}103) \\ -1{,}0 - (-1{,}045) \end{bmatrix} = \begin{bmatrix} 1{,}718 \\ -4{,}897 \\ 0{,}045 \end{bmatrix}$$

$max(|\Delta P|, |\Delta Q|) = 4{,}897 > \varepsilon$: Since the required precision is not met, the new solution is calculated.

$$J^0 = \begin{bmatrix} 43{,}89 & -21{,}95 & -2{,}160 \\ -21{,}95 & 37{,}62 & 3{,}497 \\ 2{,}160 & -3{,}702 & 35{,}53 \end{bmatrix}$$

The increments are obtained by solving the linear system:

$$\begin{bmatrix} 43{,}89 & -21{,}95 & -2{,}160 \\ -21{,}95 & 37{,}62 & 3{,}497 \\ 2{,}160 & -3{,}702 & 35{,}53 \end{bmatrix} \begin{bmatrix} \Delta\delta_2 \\ \Delta\delta_3 \\ \Delta V_3 \end{bmatrix} = \begin{bmatrix} 1{,}718 \\ -4{,}897 \\ 0{,}045 \end{bmatrix}$$

Hence:

$$\Delta x^0 = \begin{bmatrix} \Delta{\delta_2}^0 \\ \Delta{\delta_3}^0 \\ \Delta{V_3}^0 \end{bmatrix} = \begin{bmatrix} -0{,}0345 \\ -0{,}1492 \\ -0{,}0122 \end{bmatrix}$$

Solution update :

$$x^1 = x^0 + \Delta x^0 = \begin{bmatrix} 0 \\ 0 \\ 1{,}0 \end{bmatrix} + \begin{bmatrix} -0{,}0345 \\ -0{,}1492 \\ -0{,}0122 \end{bmatrix} = \begin{bmatrix} -0{,}0345 \\ -0{,}1492 \\ 0{,}9878 \end{bmatrix}$$

Convergence is reached after 4 iterations:

$$\boldsymbol{x} = \boldsymbol{x}^{4} = \begin{bmatrix} -2,1° \\ -8,8° \\ 0,98 \end{bmatrix}$$

And so:

$$\boldsymbol{V} = \begin{bmatrix} V_1\angle\delta_1 \\ V_2\angle\delta_2 \\ V_3\angle\delta_3 \end{bmatrix} = \begin{bmatrix} 1,0\angle 0° \\ 1,05\angle -2,1° \\ 0,98\angle -8,8° \end{bmatrix}$$

Injected power :

$$\boldsymbol{S} = \begin{bmatrix} \overline{S_1} \\ \overline{S_2} \\ \overline{S_3} \end{bmatrix} = \begin{bmatrix} 3,08 - j0,82 \\ 2,0 - j2,67 \\ -5,0 - j1,0 \end{bmatrix}$$

3.4 Decoupled power flow method

Power transmission lines have a very low R/X ratio. For such systems, the variation in active power is more sensitive to the variation in phase δ, while the variation in reactive power is more sensitive to changes in voltage amplitudes. It is therefore reasonable to neglect the elements in sub-matrices J_{12} et J_{21}. The system becomes:

$$\begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} \Delta\delta \\ \Delta V \end{bmatrix} \approx \begin{bmatrix} J_{11} & 0 \\ 0 & J_{22} \end{bmatrix} \begin{bmatrix} \Delta\delta \\ \Delta V \end{bmatrix} \tag{50}$$

or

$$\Delta P = J_{11}.\Delta\delta \tag{51}$$

$$\Delta Q = J_{22}.\Delta V \tag{52}$$

These are the decoupled power flow equations and are solved separately (saving time and memory space).

In the literature on the power flow problem, it is common to denote J_{11} et J_{22} as H and L. For voltage-controlled buses, the voltage amplitudes are known. Therefore, if there are m voltage-controlled buses, then H is of order $(n-1) \times (n-1)$ and L is of order $(n-1-m) \times (n-1-m)$. The elements of these two matrices are given by:

$$H_{ij} = \frac{\partial P_i}{\partial \delta_j} = V_i V_j \left(G_{ij} sin\delta_{ij} - B_{ij} cos\delta_{ij} \right) \tag{53}$$

$$H_{ii} = \frac{\partial P_i}{\partial \delta_i} = -Q_i - B_{ii} V_i^2 \tag{54}$$

$$L_{ij} = \frac{\partial Q_i}{\partial V_j} = V_i \left(G_{ij} sin\delta_{ij} - B_{ij} cos\delta_{ij} \right) \tag{55}$$

$$L_{ii} = \frac{\partial Q_i}{\partial V_i} = \frac{Q_i}{V_i} - B_{ii} V_i \tag{56}$$

3.5 Fast decoupled method

For practical electrical networks, we generally have:

$$X \gg R \quad \rightarrow \quad G_{ij} sin\delta_{ij} \ll B_{ij} \tag{57}$$

Between two adjacent buses:

$$sin\delta_{ij} = \sin(\delta_i - \delta_j) \approx \delta_i - \delta_j = \delta_{ij} \tag{58}$$

$$cos\delta_{ij} = \cos(\delta_i - \delta_j) \approx 1 \tag{59}$$

Hence:

$$Q_i = V_i \sum_{j=1}^{n} V_j (G_{ij} sin\delta_{ij} - B_{ij} cos\delta_{ij})$$

$$Q_i = V_i \sum_{j=1}^{n} V_j (G_{ij} sin\delta_{ij} - B_{ij} cos\delta_{ij}) \approx V_i \sum_{j=1}^{n} V_j (G_{ij} sin\delta_{ij} - B_{ij}) \ll V_i \sum_{j=1}^{n} V_j (-B_{ij}) \ll B_{ii}.V_i^2 \tag{60}$$

With approximations $V_i V_j \approx V_i^2$ et $V_i^2 \approx V_i$, we obtain:

$$H_{ii} = \frac{\partial P_i}{\partial \delta_i} = -V_i B_{ii} \tag{61}$$

$$H_{ij} = \frac{\partial P_i}{\partial \delta_j} = -V_i B_{ij} \tag{62}$$

Similarly, we can deduce the simplifications for the matrix L:

$$L_{ii} = \frac{\partial Q_i}{\partial V_i} = -V_i B_{ii} \tag{63}$$

$$L_{ij} = \frac{\partial Q_i}{\partial V_j} = -V_i B_{ij} \tag{64}$$

So, the decoupled equations take the following form:

$$\frac{\Delta P}{V} = -B'.\Delta\theta \tag{65}$$

$$\frac{\Delta Q}{V} = -B''.\Delta V \tag{66}$$

B' and B'' are the imaginary parts of the nodal admittance matrix Y. They are constant matrices and are inverted only once in the computation routine. The decoupled and fast decoupled methods require more iterations than the global Newton-Raphson method but much less computation time per iteration.

4 Calculation of power transmission and losses in lines and transformers

4.1 Power transmitted through the lines

To calculate the power flowing in the branch connecting buses i and j, we determine the current flowing between these buses, which is given by the following expression:

$$\bar{I}_{ij} = Y_{ij}(\bar{V}_i - \bar{V}_j) + Y_{i0}\bar{V}_i \tag{67}$$

and the complex power that passes through the $i - j$ line measured at bus i:

$$\bar{S}_{ij} = \bar{V}_i \bar{I}_{ij}^* = \bar{V}_i \cdot [Y_{ij}(\bar{V}_i - \bar{V}_j) + Y_{i0}\bar{V}_i]^* \tag{68}$$

$$\bar{S}_{ij} = |\bar{V}_i|^2 \cdot {Y_{ij}}^* - \bar{V}_i \cdot {\bar{V}_j}^* \cdot {Y_{ij}}^* + |\bar{V}_i|^2 \cdot {Y_{i0}}^* \tag{69}$$

Similarly, the apparent power flowing from bus j to bus i is:

$$\bar{S}_{ji} = \left|\bar{V}_j\right|^2 \cdot Y_{ij}{}^* - \bar{V}_j \cdot \bar{V}_i{}^* \cdot Y_{ij}{}^* + \left|\bar{V}_j\right|^2 \cdot Y_{j0}{}^* \tag{70}$$

4.2 Power transmitted through regulating transformers

Considering a regulating transformer whose admittance matrix is:

$$[Y] = \begin{bmatrix} \frac{Y_{ij}}{a^2} & -\frac{Y_{ij}}{a} \\ -\frac{Y_{ij}}{a} & Y_{ij} \end{bmatrix} \tag{71}$$

The current flowing between buses i and j is given as follows:

$$\bar{I}_{ij} = \frac{Y_{ij}}{a^2} \cdot \bar{V}_i - \frac{Y_{ij}}{a} \cdot \bar{V}_j = \frac{1}{a} \cdot Y_{ij} \left(\frac{1}{a}\bar{V}_i - \bar{V}_j\right) \tag{72}$$

and power:

$$\bar{S}_{ij} = \bar{V}_i \bar{I}_{ij}^* = \bar{V}_i \cdot \left[\frac{1}{a} \cdot Y_{ij} \left(\frac{1}{a}\bar{V}_i - \bar{V}_j\right)\right]^* = \frac{1}{a^2} \cdot |\bar{V}_i|^2 \cdot Y_{ij}^* - \frac{1}{a} \cdot Y_{ij}{}^* \bar{V}_j{}^* \cdot \bar{V}_i \tag{73}$$

On the other hand, the current flowing from bus j to bus i:

$$\bar{I}_{ji} = \frac{1}{a} \cdot Y_{ij} \left(\bar{V}_j - \frac{1}{a}\bar{V}_i\right) \tag{74}$$

$$\bar{S}_{ji} = \bar{V}_j \bar{I}_{ji}^* = \bar{V}_j \cdot \left[\frac{1}{a} \cdot Y_{ij} \left(\bar{V}_j - \frac{1}{a}\bar{V}_i\right)\right]^* = \frac{1}{a^2} \cdot \left|\bar{V}_j\right|^2 \cdot Y_{ij}^* - \frac{1}{a} \cdot Y_{ij}{}^* \bar{V}_i{}^* \cdot \bar{V}_j \tag{75}$$

4.3 Calculation of total losses in the network

For an element of the network connecting two buses i and j, we have:

$$\bar{S}_{Loss\,ij} = \bar{S}_{ij} + \bar{S}_{ji} \tag{76}$$

$\bar{S}_{Loss\,ij}$: complex power consumed (lost) in the branch $(i - j)$
$\bar{S}_{ij}$: complex power that passes from bus i to bus j
$\bar{S}_{ij}$: complex power that flows from bus i to bus j.

$$P_{Loss\,ij} = \mathcal{R}eel\{\bar{S}_{Loss\,ij}\} \tag{77}$$

$$Q_{Loss\,ij} = \mathcal{I}mag\{\bar{S}_{Loss\,ij}\} \tag{78}$$

$P_{Loss\,ij}$: active power lost in the branch $(i - j)$
$Q_{Loss\,ij}$: reactive power lost in the branch $(i - j)$
The total power lost in the network is equal to the sum of the power lost in all branches of the network.

$$\bar{S}_{Loss} = \sum \bar{S}_{Loss\,ij} \tag{79}$$

$$P_{Loss} = \mathcal{R}eel\left\{\sum \bar{S}_{Loss\,ij}\right\} \tag{80}$$

$$Q_{Loss} = \mathcal{I}mag\left\{\sum \bar{S}_{Loss\,ij}\right\} \tag{81}$$

5 Representation of power flow results

Consider the network in Fig. 6 below.

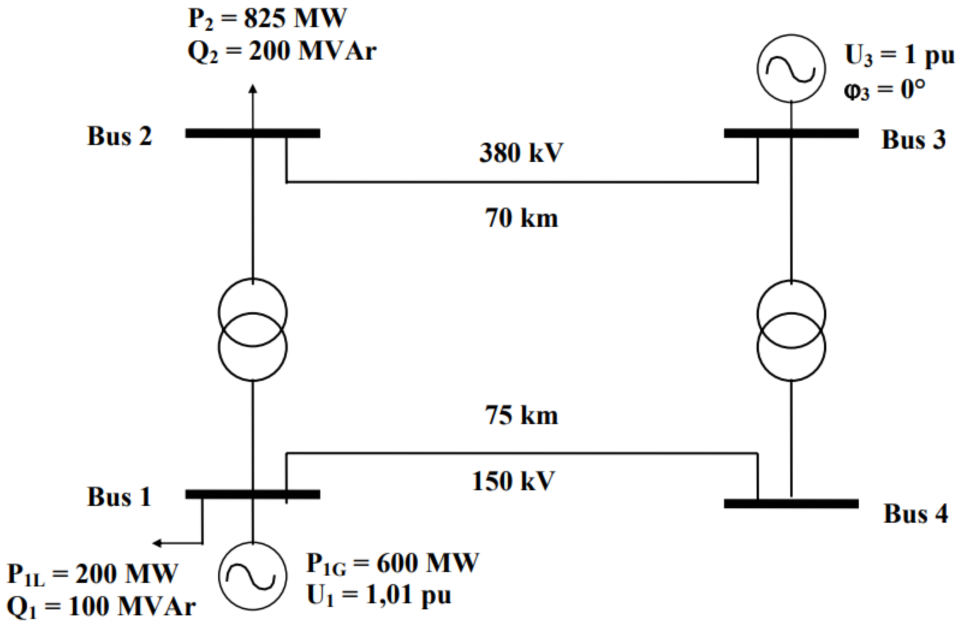

Fig. 6: 4-bus network

The data for this network are shown in the following tables.

Lines

Line	i	j	R (Ω/km)	X (Ω/km)	U (kV)	l (km)
1	1	4	0	0.2875	150	75
2	2	3	0	0.2100	380	70

Transformers

Transfo	i	j	R (%)	X (%)	S (MVA)
1	1	2	0	13	295
2	3	4	0	13	295

Buses

Bus	Type	U (pu)	P_G (MW)	Q_G $(MVAr)$	$P_L(MW)$	$Q_L(MVAr)$
1	PV	1.01	600	0	200	100
2	PQ	-	-	-	825	200
3	$Slack$	1.00	0	0	0	0
4	PQ	-	-	-	0	0

Power flow results

The results of the load flow calculation applied to this network are listed below. This represents the electrical state of the network for the imposed constraints (PV, PQ, and slack buses; scheduled generated and consumed powers). For each bus, the first line gives the nominal voltage (kV), the current voltage (pu), and the phase shift with respect to the slack bus. The other lines represent the connected loads and generators, as well as the power flows in the lines and transformers. We can directly verify that the sum of active or reactive powers entering a bus is zero.

```
BUS       1 Bus 1     150.0    MW      Mvar      MVA    % 1.0100    5.52    1 1
 GENERATOR 1               600.00   199.36R   632.3
 LOAD 1                    200.00   100.00    223.6
 TO       2 Bus 2      1   330.65    88.79    342.4  34   1.0000NT    0.0
 TO       4 Bus 4      1    69.41    10.56     70.2   7

BUS       2 Bus 2     380.0    MW      Mvar      MVA    % 0.9819   -2.94    1 1
 LOAD 1                    825.00   200.00    848.9
 SWITCHED SHUNT              0.00     0.00      0.0
 TO       1 Bus 1      1 -330.65   -38.12    332.8  33   1.0000TA    0.0
 TO       3 Bus 3      1 -494.35 -161.88    520.2   5

BUS       3 Bus 3     380.0    MW      Mvar      MVA    % 1.0000    0.00    1 1
 GENERATOR 1               424.94   186.65R   464.1
 TO       2 Bus 2      1   494.35   190.45    529.8   5
 TO       4 Bus 4      1   -69.41    -3.80     69.5   7   1.0000TA    0.0

BUS       4 Bus 4     150.0    MW      Mvar      MVA    % 1.0021    1.75    1 1
 TO       1 Bus 1      1   -69.41    -5.93     69.7   7
 TO       3 Bus 3      1    69.41     5.93     69.7   7   1.0000NT    0.0
```

Graphical representation

Graphical interfaces make it easier to visualize the electrical quantities of the network and to monitor its state. One example of such a representation is provided by the PowerWorld software: power transfers are shown by arrows along the lines, and at the buses the voltages and injected powers are displayed (Fig. 7).

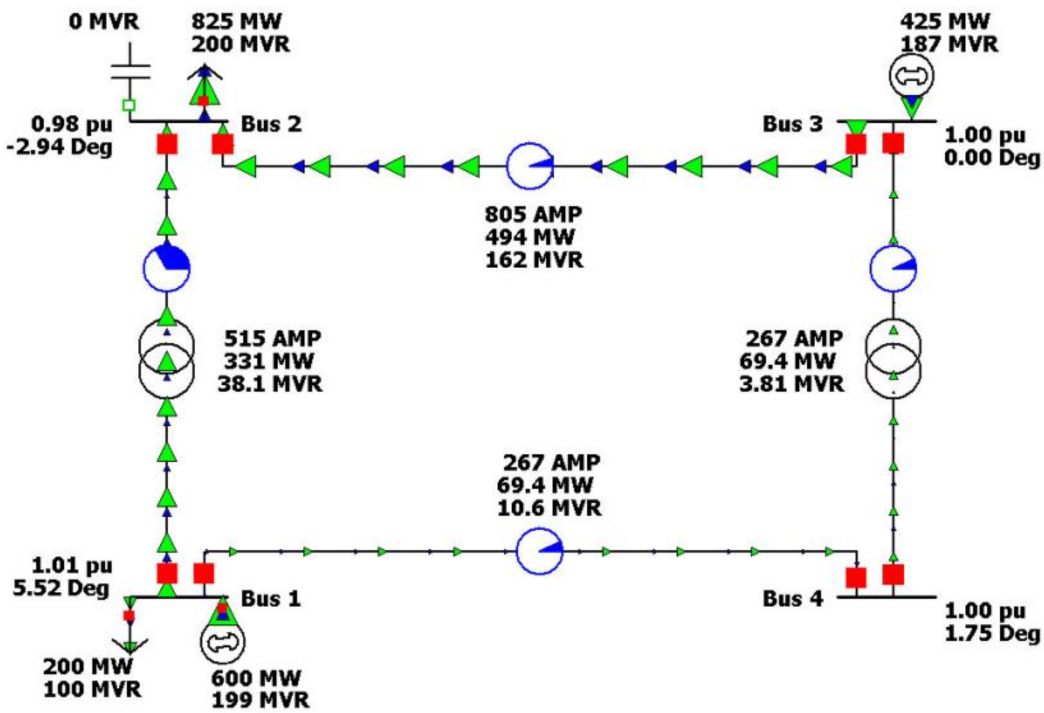

Fig. 7: Example of a power flow presentation