OPTIMAL POWER DISPATCH or ECONOMIC DISPATCH

1 Introduction

Electricity is not available in nature and can be stored only in small quantities; it must therefore be produced, transmitted, and distributed continuously, while in particular satisfying the technical conditions of stable voltage and frequency. For this reason, during the operation of the power system, the contribution of each power plant and each generator, so as to satisfy the demand of consumers for electrical energy at every time and place, must be determined in such a way that the cost of energy production is minimal. This problem is known, from the economic point of view, as 'Optimal Economic Dispatch'.

Initially, the operating method consisted of loading, or making generate at the maximum, the most efficient units. This solution is not profitable because overuse of the machines reduces their service life. Consequently, servicing and maintenance costs increased considerably. The expansion and complexity of the network have led researchers to adopt other methods to help alleviate this problem.

2 Purpose of optimal economic dispatch

Large-scale storage of electrical energy in a form that is immediately available is not currently possible under satisfactory consumption conditions. The major operational problem is to maintain, at all times, the instantaneous balance between generation and consumption. This is a necessary condition for the system that ensures the production and delivery of electrical energy to consumers.

Electricity is produced in power stations by means of hydro- or steam-driven turbo-alternators. Most of the world’s electricity is produced in hydroelectric and thermal power plants using different fuels (coal, gas, nuclear energy, etc.), with smaller shares from diesel and other internal-combustion installations.

The operation and control of an electrical network pursue the following objectives:

- Promote economic performance: overall optimization taking into account the relationship between production cost and transmission cost, which essentially amounts to minimizing the production cost at all times.
- Maintain the operational security of the system: ensuring, at every instant, the supply of consumers and compliance with rules that prevent the risk of a general collapse.
- Guarantee quality of supply.

The objective of optimal economic dispatch is therefore to minimize the production cost of electrical energy while satisfying equality and inequality constraints. A few examples are given below:

- Economic dispatch tends to minimize the quantity of water wasted or to produce the maximum MWh from the quantity of water available in a network composed of hydroelectric plants.
- In a network composed of coal-fired thermal power plants, economic dispatch aims to allocate generation among the different stations so as to minimize the pollution level.

- Consider two generators, one operating on oil and the other on gas. In this case, economic dispatch makes it possible to share the load between the two generators so that the overall production cost is minimal.

In Algeria, since the fuel used is mainly natural gas, the aim is to optimize the specific consumption of gas- and steam-turbine generators.

3 Problem formulation

Generating units are the essential components of a given electrical network. The task is to satisfy load demand and cover power losses at minimum cost.

The problem consists in minimizing the fuel-cost function for the production of the required electrical energy. This function defines the relationship between generator production costs and generated powers. This optimization problem is generally formulated as follows:

$$\begin{array}{ll} \text{Minimize the objective function} & F(X) \\ \text{Subject to the equality constraints} & G(X) = 0 \\ \text{and the inequality constraints} & H(X) \leq 0 \end{array}$$

The equality constraints define the power-flow equations or the balance equation between demand and generation. The inequality constraints, on the other hand, define the admissible operating ranges and essentially resource limitations, security margins, etc.

The objective function is based on economic criteria represented by production costs and line power losses. The total cost function is the sum of the cost functions of all generating units. This cost function is generally considered to be a quadratic polynomial function.

4 Characteristics of generated power

In analyzing the optimization problem of the cost function of an electric power system, several parameters are taken into account. Indeed, the economic dispatch problem is the determination of generation levels such that the total production cost becomes minimum for a specified load level. The most important parameters are the input-output characteristics of the generating units. For thermal units, the fuel cost per unit of power varies considerably with the output power of the unit. Consequently, the fuel-cost characteristics of the generators must be taken into account while determining their optimal real powers. Fig. 1 shows a simplified diagram of a steam power plant. This unit consists of a boiler, a turbine, and a generator.

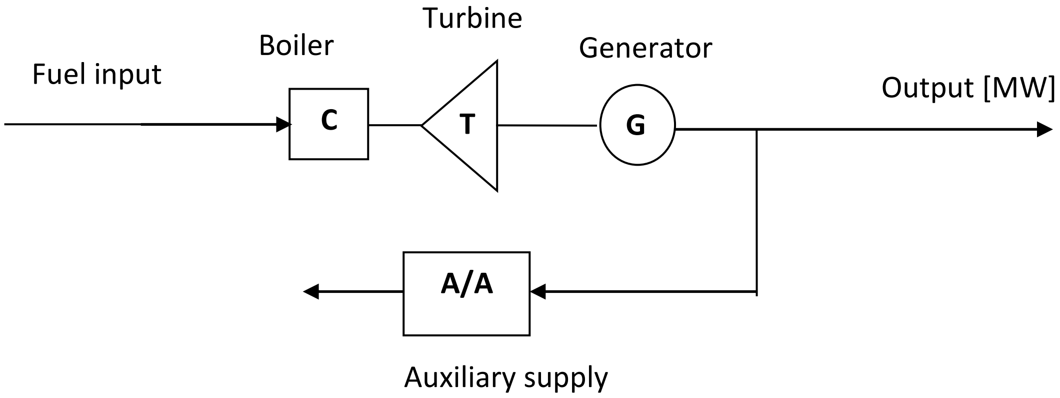

Fig. 1: Steam thermal power plant

In general, labor and maintenance costs are fixed. Part of the gross power applied to the turbine supplies, at the generator output, the auxiliaries such as the boiler-feed-system pumps, cooling-water pumps, etc. The quantity of gross energy is characterized either by the unit of fuel consumption or by the price per hour at the input, and by the amount of electrical energy in MWh at the output. The variation of the fuel quantity $H_i\ (P_i)$ or of its cost $F_i\ (P_i)$ at the input as a function of the net power generated at the plant output is called the input-output characteristic. In practice, it is obtained from a discrete number of points provided by a series of tests on the generating unit. Fig. 2 shows a usual form of the input-output characteristic, a quadratic function of the output power. P_{min} is the output level for which operation of the unit is neither economically nor technically feasible. P_{max} is the maximum output limit.

For the calculation of the optimal cost of produced energy, these characteristics are represented by quadratic functions of the form:

$$F(P) = a + bP + cP^2 \tag{1}$$

Where the coefficients a, b et c are specific to each generating unit.

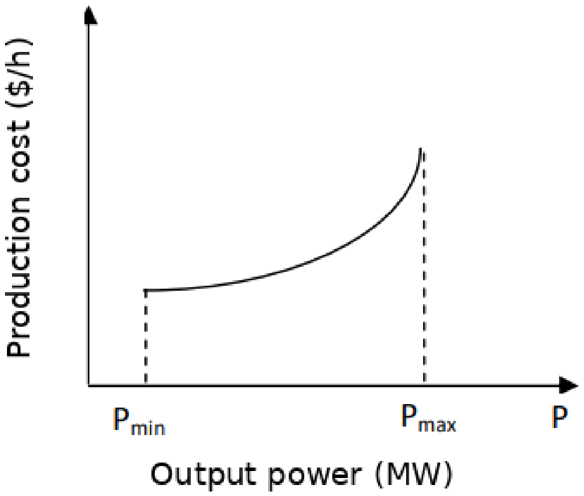

Fig. 2: Input-output characteristic curve of a steam thermal unit

5 Economic dispatch without consideration of transmission losses

Fig. 3 shows the configuration studied in this type of problem. This system contains N generators, or generating units, connected to a load that requires a total demanded power P_D. The generator inputs are F_i, which represent fuel costs. The outputs are the generated powers P_i.

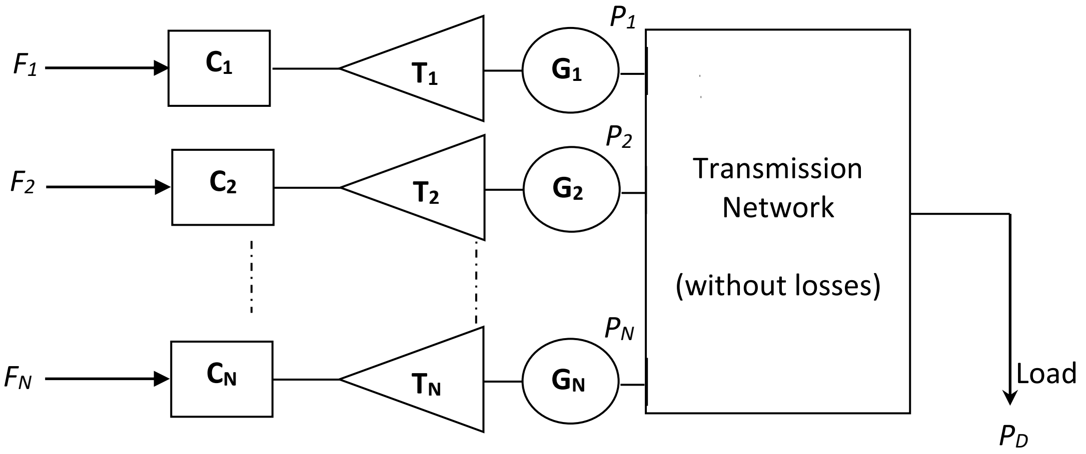

Fig. 3: Connection of power plants to a lossless transmission network

5.1 Cost function

Reactive-power generation has no concrete effect on the cost function because it is controlled by variations in the excitation current of the generators. Thus, the total cost function F_T is expressed only as a function of the generated powers P_i.

$$F_T = \sum_{i=1}^{N} F_i(P_i) = F_1(P_1) + F_2(P_2) + \cdots + F_N(P_N) \tag{2}$$

where N is the number of generating units.

F_i is the cost, expressed in [$/MWh], specific to energy production in generating unit number i.

A cost function is said to be separable if it can be written as the sum of terms each depending on a single variable.

5.2 Incremental cost "IC"

The incremental cost of generation for the controllable generators of unit i is defined by:

$$IC_i = \frac{\partial F_i}{\partial P_i} \tag{3}$$

Fig. 4 shows the cost characteristic as a function of the generated power P_i. The incremental cost of unit i is the partial derivative $\frac{\partial F_i}{\partial P_i}$, which represents the tangent to the curve relating cost to generated power.

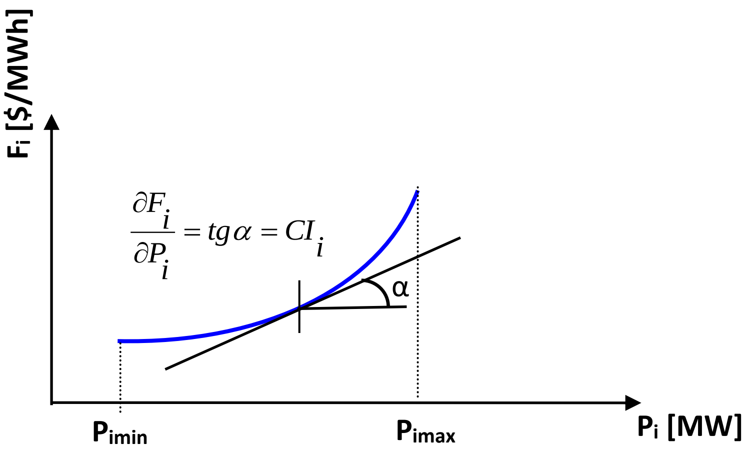

Fig. 4: Cost function and incremental cost

5.3 Constraints

A set of variables P_i must be chosen so as to minimize the cost function as much as possible. This choice cannot be random, because the equality and inequality constraints must be satisfied.

1) Equality constraint

Neglecting transmission losses, total generation must balance the total load. Energy balance is obtained when the variables P_i satisfy:

$$\sum_{i=1}^{N} P_i = P_D \tag{4}$$

The demanded power is considered constant because demand varies relatively slowly over periods of 2 to 3 minutes.

2) Inequality constraints

Each generator must not operate above its rated power or below a certain minimum value. These conditions are expressed mathematically by the following inequality:

$$P_{imin} \leq P_i \leq P_{imax} \quad \text{avec} \quad i = 1 \cdots N \tag{5}$$

5.4 Lagrangian formulation

The problem is to choose the generator outputs $P_1 \dots P_N$ such that the cost function F_T is minimized while satisfying the equality constraint (4) and the inequality constraints (5). Thus, the dispatching problem becomes a constrained nonlinear optimization problem. It can be solved by developing a function called the Lagrangian function (the "Kuhn-Tucker" method).

To obtain an extremum of an objective function under constraints, the constraint function must be added to the objective function and multiplied by a coefficient called the 'Lagrange multiplier', which is initially undetermined. The augmented Lagrangian function of the problem is given by:

$$\mathcal{L} = F_T + \lambda\phi \quad \text{avec } \phi = P_D - \sum_{i=1}^{N} P_i = 0 \tag{6}$$

The necessary condition for obtaining the optimum is that the first derivatives of the Lagrangian function with respect to P_i and λ be equal to zero. In this case, there are $N+1$ variables whose unknowns are the N generated powers and the Lagrange multiplier λ. The derivative of the Lagrangian function with respect to λ yields only the equality constraint. In other words, the optimal generated powers are obtained when the derivatives of the cost function with respect to the generated powers are equal to zero, while ensuring that their sum is equal to the total demanded power (known as the Karush-Kuhn-Tucker optimality conditions, KKT, or more simply Kuhn-Tucker, KT). The first derivatives of equation (6) are given by :

$$\frac{\partial \mathcal{L}}{\partial P_i} = \frac{\partial F_T}{\partial P_i} - \lambda = 0 \tag{7}$$

Consequently, in simplified form, equation (7) becomes:

$$\frac{\partial F_i}{\partial P_i} = \lambda = IC_i \tag{8}$$

The condition for the existence of an optimum for the cost function of the generating units is that the IC_i of each unit i be equal, for every generator, to the same previously undetermined value, namely λ. The inequality constraint is that the generated powers must not exceed their limits. This problem can be summarized as follows:

$$\begin{aligned} &\frac{\partial F_i}{\partial P_i} = \lambda && N \text{ equations} \\ &P_{imin} \le P_i \le P_{imax} && 2N \text{ inequalities} \\ &\sum_{i=1}^{N} P_i = P_D && 1 \text{ equation} \end{aligned} \tag{9}$$

Thus, the variation of the incremental cost will depend on $[P_{imin}, P_{imax}]$ and will be governed by the equation:

$$\begin{aligned} &\frac{\partial F_i}{\partial P_i} = \lambda && \text{for} && P_{imin} \le P_i \le P_{imax} \\ &\frac{\partial F_i}{\partial P_i} \le \lambda && \text{for} && P_{imax} = P_i \\ &\frac{\partial F_i}{\partial P_i} \ge \lambda && \text{for} && P_{imin} = P_i \end{aligned} \tag{10}$$

Considering the quadratic form (1) of the cost function, we obtain:

$$\frac{\partial F_i}{\partial P_i} = b_i + 2c_i P_i = \lambda \tag{11}$$

or

$$P_i = \frac{\lambda - b_i}{2c_i} \tag{12}$$

Substituting into (4) :

$$\sum_{i=1}^{N} \frac{\lambda - b_i}{2c_i} = P_D \tag{13}$$

Hence:

$$\lambda = \frac{P_D + \sum_{i=1}^{N} \frac{(b_i}{2c_i)}}{\sum_{i=1}^{N} \frac{1}{2c_i}} \tag{14}$$

Having λ, the values of P_i can be obtained from (12).

Example:

Two generating units of an electrical system have the following cost curves:

$$F_1(P_1) = 120 + 22P_1 + 0{,}05P_1^2 \qquad (\$/h)$$

$$F_2(P_2) = 120 + 16P_2 + 0{,}06P_2^2 \qquad (\$/h)$$

P_1 and P_2 are given in MW. The two units operate continuously, and their generation limits are 100 MW and 20 MW. Determine the economic power dispatch for a total load of 80 MW, neglecting transmission losses.

Using (14):

$$\lambda = \frac{80 + \left(\frac{22}{2 \times 0{,}05}\right) + \left(\frac{16}{2 \times 0{,}06}\right)}{\left(\frac{1}{2 \times 0{,}05}\right) + \left(\frac{1}{2 \times 0{,}06}\right)} = 23{,}64\ \$/MWh$$

Then, using (12):

$$P_1 = \frac{23{,}64 - 22}{2 \times 0{,}05} = 16{,}36\ MW$$

$$P_2 = \frac{23{,}64 - 16}{2 \times 0{,}06} = 63{,}64\ MW$$

But $P_{1min} = 20\ MW$, therefore P_1 must be fixed at 20 MW and the rest of the load is supplied by P_2.

$$P_1 = 20\ MW \quad \text{and} \quad P_2 = 80 - 20 = 60\ MW$$

6 Economic dispatch considering losses

When it is necessary to transmit electrical energy over long distances or when supplying a large area with a relatively low concentration of load, the influence of transmission losses on the economic dispatch problem must be taken into account. The configuration is the same as before, but the transmission network is subject to power losses.

If losses are taken into account in the power balances, the problem becomes more difficult because the equality constraint contains the total active transmission losses P_L.

The power-balance equation taking losses into account is given by:

$$P_D + P_L - \sum_{i=1}^{N} P_i = \phi = 0 \tag{15}$$

Active losses are functions of the impedance of the electrical network and the line current in the transmission lines. Consequently, the electric current is linked only to the generated powers and the demanded powers.

The Lagrangian function for this type of problem becomes:

$$\mathcal{L} = F_T + \lambda\phi \quad \text{avec } \phi = P_D + P_L - \sum_{i=1}^{N} P_i \tag{16}$$

The condition for minimizing the cost function is:

$$\frac{\partial \mathcal{L}}{\partial P_i} = \frac{\partial F_T}{\partial P_i} - \lambda\left(1 - \frac{\partial P_L}{\partial P_i}\right) = 0 \tag{17}$$

Or:

$$\frac{\partial F_T}{\partial P_i} + \lambda\frac{\partial P_L}{\partial P_i} = \lambda \tag{18}$$

From equation (18), we obtain:

$$\lambda = \frac{\partial F_i}{\partial P_i} \times \frac{1}{\left(1 - \frac{\partial P_L}{\partial P_i}\right)} \tag{19}$$

Therefore:

$$\lambda = IC_i \times \frac{1}{(1 - IPT_i)} \quad \text{with } IPT_i = \frac{\partial P_L}{\partial P_i} \tag{20}$$

IPT_i : Incremental transmission losses.

This last equation is the necessary condition for the existence of a minimum operating cost. The incremental cost of all units is the same. To this condition must be added the constraints on the powers.

The exact value of transmission losses can only be obtained from a power-flow study. Nevertheless, in economic dispatch studies, transmission losses are often expressed as a function of the generated active powers. This technique is commonly called the B-coefficient method, which is used to estimate these losses.

In this approach, losses are approximated by Kron’s formula:

$$P_L = \sum_{i=1}^{N}\sum_{j=1}^{N} P_i B_{ij} P_j + \sum_{i=1}^{N} B_{0i} P_i + B_{00} \tag{21}$$

The coefficients B_{ij} are called loss coefficients or coefficients B.

The derivative of the Lagrangian function then becomes:

$$\frac{\partial \mathcal{L}}{\partial P_i} = \frac{\partial F_T}{\partial P_i} - \lambda\left(1 - 2\sum_{j=1}^{N} B_{ij} P_j - B_{i0}\right) \tag{22}$$

In simplified form, with $B_{ij} = B_{00} = 0\ (i \neq j)$, equation (21) can be reduced to:

$$P_L = \sum_{i=1}^{N}\left(B_{ii} {P_i}^2\right) \tag{23}$$

7 Solution by the iterative lambda method

This method is based on an initial estimate of the incremental cost. Graphically, Fig. 5 shows its principle. The optimal powers are determined so as to minimize the total cost function. The first approach is to draw horizontal incremental-cost lines for each generator in the same plane. To obtain the optimum and at the same time satisfy the total demand, an incremental cost value λ is estimated on the graph and the generated powers of each unit corresponding to this value of λ are found. Their sum must be compared with the total demanded power PD plus the transmission losses P_L. For the second estimate of λ, λ must be increased or decreased depending on whether the sum is less than or greater than the total demanded power P_D plus the transmission losses P_L. If some generators exceed the limit, that limit must be adopted and the calculation process continued for the others.

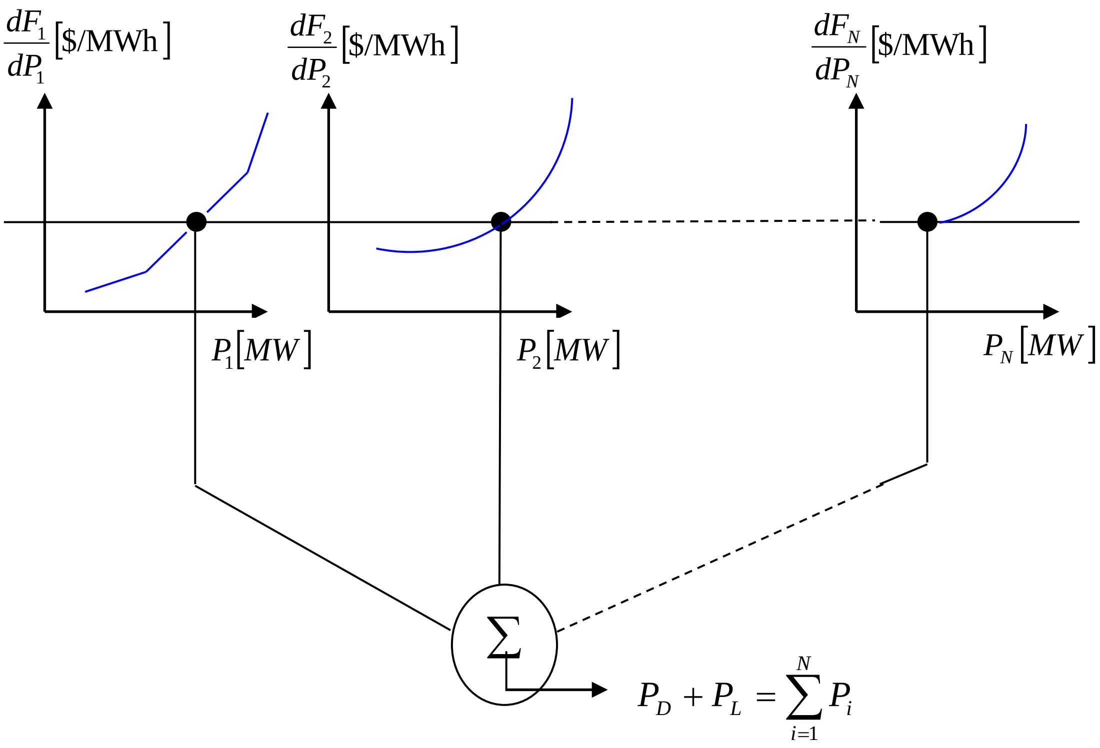

Fig. 5: Graphical solution

With two solutions, we can extrapolate or interpolate between them to get closer to the desired value of the total generated power, as shown in Fig. 6. With this method, convergence is very fast.

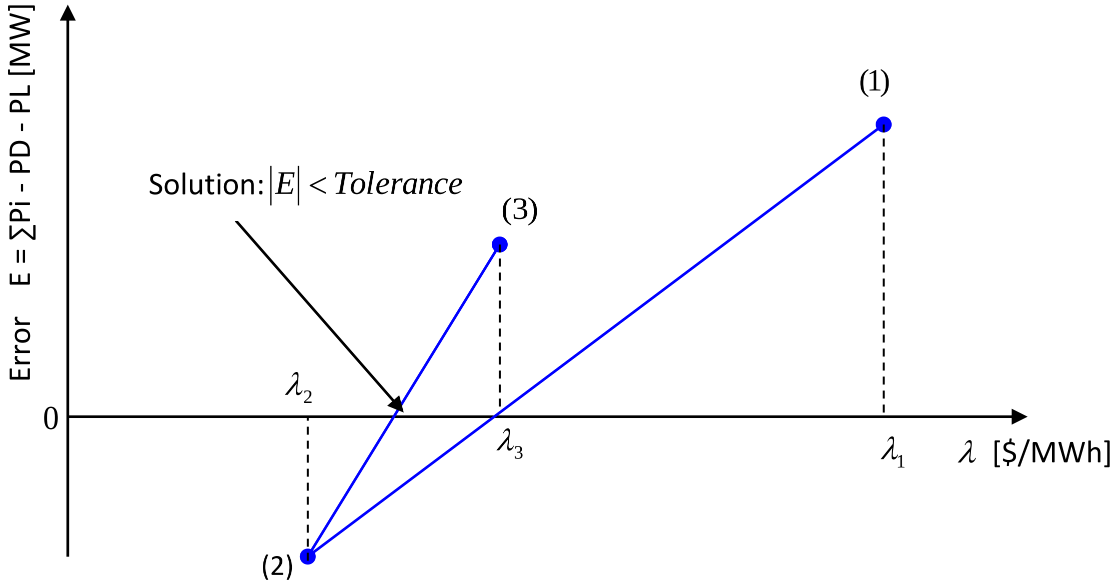

Fig. 6: Lambda projection

The technique most commonly used to reach λ_{opt} quickly is the Newton-Raphson method. Applying the Newton-Raphson method makes it possible to solve the equation:

$$E = 0 = f(\lambda) = \sum_{i=1}^{N} P_i - P_D - P_L \tag{24}$$

Using the quadratic form of the cost function given by (1) and the transmission losses formulated by (23), the incremental cost $\frac{\partial F_i}{\partial P_i}$ of each unit i can be deduced.

$$\frac{\partial F_i}{\partial P_i} = \lambda\left(1 - \frac{\partial P_L}{\partial P_i}\right) = b_i + 2c_i P_i \tag{25}$$

Therefore:

$$P_i = \frac{\lambda\left(1 - \frac{\partial P_L}{\partial P_i}\right) - b_i}{2c_i} = \frac{\lambda(1 - 2B_{ii}P_i) - b_i}{2c_i} \tag{26}$$

Using (24), we can write:

$$E(\lambda) = \sum_{i=1}^{N} P_i - P_D - P_L = \sum_{i=1}^{N} \frac{\lambda\left(1 - \frac{\partial P_L}{\partial P_i}\right) - b_i}{2c_i} - P_D - P_L \tag{27}$$

The Newton-Raphson algorithm then gives:

$$\lambda^{it+1} = \lambda^{it} - \frac{E(\lambda^{it})}{E'(\lambda^{it})} = \frac{P_D + P_L + \sum_{i=1}^{N} \frac{b_i}{2c_i}}{\sum_{i=1}^{N} \frac{1 - 2B_{ii}P_i}{2c_i}} \tag{28}$$

This technique converges very quickly to the solution and allows the calculation of the generated active powers. The initial value of λ must lie between λ_{min} and λ_{max}, corresponding respectively to P_{imin} and P_{imax}.

Calculation algorithm

Step 1: Data for a_i, b_i, c_i, P_{0i} et B_{ii}

Step 2: Choose an initial value λ_0

Step 3: Calculate the generated powers

$$P_i = \frac{\lambda(1 - 2B_{ii}P_i) - b_i}{2c_i}$$

Step 4: Calculate the transmission losses

$$P_L = \sum_{i=1}^{N} \left(B_{ii} {P_i}^2\right)$$

Step 5: If $P_i < P_{imin}$, set $P_i = P_{imin}$
If $P_i > P_{imax}$, set $P_i = P_{imax}$

Step 6: Calculate the sum of the generated powers.

Step 7: Calculate the error

$$E=\sum_{i=1}^{N} P_i - P_D - P_L$$

Step 8: If $|E| < \varepsilon$, go to Step 10. Otherwise, continue to Step 9.

Step 9: Calculate λ by

$$\lambda=\frac{P_D+P_L+\sum_{i=1}^{N}\frac{b_i}{2c_i}}{\sum_{i=1}^{N}\frac{1-2B_{ii}P_i}{2c_i}}$$

Return to Step 3

Step 10: Display the generated powers and the total cost.

Step 11: End

8 Solution by the first-order gradient method

This is a direct enumeration method that uses the behavior of the objective function as the independent variable. It starts from a feasible solution and then searches for the optimal solution along a trajectory that keeps the solution feasible at all times.

Equation (29) represents the Taylor expansion of the expression F_T and its interpolation at feasible operating points. For the first order, the change in the objective function is given in equation (30). This relation is obtained if and only if the Taylor-series terms of order higher than 1 can be neglected with respect to the first-order term. This is possible only by subtracting the initial operating cost from the perturbed equation.

$$F_T+\Delta F_T=F_1(P_1)+F_2(P_2)+\cdots+F_N(P_N)+\frac{dF_1}{dP_1}\Delta P_1+\frac{dF_2}{dP_2}\Delta P_2+\cdots \tag{29}$$

$$\Delta F_T=\frac{dF_1}{dP_1}\Delta P_1+\frac{dF_2}{dP_2}\Delta P_2+\cdots+\frac{dF_N}{dP_N}\Delta P_N \tag{30}$$

The equality-constraint equation is written as follows:

$$\sum_{i=1}^{N}\Delta P_i=0 \tag{31}$$

In this relation, the initial values are subtracted. Therefore, the sum of the changes in all generated powers is equal to zero. To solve the economic dispatch problem by this method, one unit must be chosen as the dependent variable. In general, the last unit is considered the dependent unit ($x=N$).

The change in the generated powers for this dependent unit is therefore a negative sum over the $N-1$ units, as shown in equation (32):

$$\Delta P_x = -\sum_{\substack{i=1 \\ i\neq x}}^{N} \Delta P_i \tag{32}$$

From equations (30) and (32), we obtain:

$$\Delta F_T = \sum_{\substack{i=1 \\ i\neq x}}^{N} \left[\frac{dF_i}{dP_i} - \frac{dF_x}{dP_x}\right] \Delta P_i = \sum_{\substack{i=1 \\ i\neq x}}^{N} \frac{\partial F_T}{\partial P_i} \Delta P_i \tag{33}$$

Calculation algorithm

Step 1: Choose a feasible P_i and calculate $\frac{dF_i}{dP_i}$ using $i = 1, \dots, N$

Step 2: Select the dependent variable $x = N$

Step 3: Calculate $\frac{dF_T}{dP_i} = \frac{dF_i}{dP_i} - \frac{dF_x}{dP_x}$ with $i = 1, \dots, N$ and $i \neq x$

Step 4: Determine the maximum derivative: $\left[\frac{\partial F_T}{\partial P_i}\right]_{max}$ and its index i_{max}

Step 5: Calculate $P_{i_{max}} = P_{i_{max}} + \Delta P$; $P_x = P_x - \Delta P$

Step 6: If $P_{i_{max}}$ is outside the limits P_{imin} and P_{imax}

Adjust $P_{i_{max}}$ and go to Step 3

Step 7: If P_x is outside the limits, adjust P_x and go to Step 2

Step 8: Calculate ΔF_T

Step 9: If $\Delta F_T > \varepsilon$, go to Step 3. Otherwise go to Step 10.

Step 10: Display the results

9 Solution by the second-order gradient method

The first-order gradient method can be effectively improved by using the second-order term of the Taylor series in the total production cost. The Taylor series is written:

$$\begin{aligned} F_T + \Delta F_T = {} & F_1(P_1) + F_2(P_2) + \dots + F_N(P_N) + \frac{dF_1}{dP_1}\Delta P_1 + \frac{dF_2}{dP_2}\Delta P_2 + \dots + \frac{dF_N}{dP_N}\Delta P_N \\ & + \frac{1}{2}\left(\frac{d^2F_1}{dP_1^2}(\Delta P_1)^2 + \frac{d^2F_2}{dP_2^2}(\Delta P_2)^2 + \dots + \frac{d^2F_N}{dP_N^2}(\Delta P_N)^2\right) + \cdots \end{aligned} \tag{34}$$

In this case, Taylor-series terms of order higher than 2 must be neglected. The second partial derivative of the units is standard and depends explicitly on the generated-power terms: $\frac{\partial^2 F_i}{\partial P_i \partial P_j} = 0$ with $i \neq j$. The constraints require that the sum of the generated powers be equal to the total demand and be treated as in equation (35):

$$\sum_{i=1}^{N} \Delta P_i = 0 \tag{35}$$

$$\Delta P_x = -\sum_{\substack{i=1 \\ i \neq x}}^{N} \Delta P_i \tag{36}$$

By substituting relation (36) into equation (34), while keeping only the first two terms, we obtain:

$$\begin{aligned} \Delta F_T = & \sum_{\substack{i=1 \\ i \neq x}}^{N} \left[\frac{dF_i}{dP_i} - \frac{dF_x}{dP_x}\right] \Delta P_i \\ & + \frac{1}{2}\left\{ \sum_{\substack{i=1 \\ i \neq x}}^{N} \left[\frac{d^2F_1}{d{P_1}^2}(\Delta P_1)^2 + \frac{d^2F_2}{d{P_2}^2}(\Delta P_2)^2 + \cdots\right] \right. \\ & \left. + \frac{d^2F_x}{d{P_x}^2}\left(\Delta {P_1}^2 + \Delta {P_2}^2 + \cdots + 2\Delta P_1 \Delta P_2 + 2\Delta P_1 \Delta P_3 + \cdots\right) \right\} \end{aligned} \tag{37}$$

The best operating point is achieved when the partial derivative $\frac{\partial \Delta F_T}{\partial \Delta P_i} = 0, \forall i$ with i ≠ x.

$$\begin{aligned} \frac{\partial \Delta \mathrm{F_T}}{\partial \Delta \mathrm{P_1}} &= 0 = \left(\frac{dF_1}{dP_1} - \frac{dF_x}{dP_x}\right) + \frac{d^2F_1}{dP_1^2}\Delta P_1 + \frac{d^2F_x}{dP_x^2}\sum_{i \neq x} \Delta P_i \\ \frac{\partial \Delta F_T}{\partial \Delta P_2} &= 0 = \left(\frac{dF_2}{dP_2} - \frac{dF_x}{dP_x}\right) + \frac{d^2F_2}{dP_2^2}\Delta P_2 + \frac{d^2F_x}{dP_x^2}\sum_{i \neq x} \Delta P_i \\ & \cdots \ldots \end{aligned} \tag{38}$$

With:

$$F_i' = \frac{dF_i}{dP_i} \quad \text{et} \quad F_i'' = \frac{d^2F_i}{d{P_i}^2} \tag{39}$$

The two derivatives can be evaluated for an initial operating point. Then the $N - 1$ equations can be written simultaneously in matrix form:

$$\begin{bmatrix} F_1'' + F_x'' & F_x'' & F_x'' & \cdots \\ F_x'' & F_2'' + F_x'' & F_x'' & \cdots \\ F_x'' & F_x'' & F_3'' + F_x'' & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} \Delta P_1 \\ \Delta P_2 \\ \Delta P_3 \\ \cdots \end{bmatrix} = - \begin{bmatrix} F_1' - F_x' \\ F_2' - F_x' \\ F_3' - F_x' \\ \cdots \end{bmatrix} \tag{40}$$

Consequently, from (40), we obtain:

$$F_x''\Delta P_1 + \cdots + F_x''\Delta P_{i-1} + (F_i'' + F_x'')\Delta P_i + F_x''\Delta P_{i+1} + \cdots + F_x''\Delta P_{i+N-1} = -(F_i' - F_x') \quad i = 1, \cdots, N \;\; \text{and} \;\; i \neq x \tag{41}$$

$$f_j(\Delta P_j) = F_j''.\Delta P_j + \sum_{\substack{i=1\\ i\neq x}}^{N-1} F_x''.\Delta P_i + \left(F_j' - F_x'\right) = 0 \tag{42}$$

$$j = 1, \cdots, N-1 \quad \text{and} \quad j \neq x$$

where $F_j = \begin{bmatrix} f_1 \\ \dots \\ f_{N-1} \end{bmatrix}$

The solution can be obtained by the Newton-Raphson algorithm with ΔP_i as the state variable.

Calculation algorithm

Step 1: Read the system data

Step 2: Form the matrices F' and F''

Step 3: Initialize ΔP_i^0

Step 4: Initialize the iteration counter $\lambda = 1, \dots, m$

Step 5: Calculate ΔP_i with $i = 1, \dots, N-1$ and $F_j = \begin{bmatrix} f_1 \\ \dots \\ f_{N-1} \end{bmatrix}$ as follows

$$J = \begin{bmatrix} \dfrac{\partial f_1}{\partial \Delta P_1} & \dots & \dfrac{\partial f_{N-1}}{\partial \Delta P_1} \\ \dots & 0 & \dots \\ \dfrac{\partial f_1}{\partial \Delta P_{N-1}} & \dots & \dfrac{\partial f_{N-1}}{\partial \Delta P_{N-1}} \end{bmatrix}$$

Step 6: If $\left|\Delta P_i^{k+1} - \Delta P_i^k\right| < \varepsilon$, go to the next step. Otherwise, go to Step 3

Step 7: Calculate $P_i = P_{i0} + \Delta P_i$ with $i = 1, \dots, N$

Step 8: Calculate $P_x = P_D - \sum_{i=1}^{N-1} P_i$ with $x = N$

10 Application example

Consider a 3-generator-bus network whose cost functions, expressed in $ per hour, are given by the following expressions:

$$F_1(P_1) = 0{,}001562P_1^2 + 7{,}92P_1 + 561$$

$$F_2(P_2) = 0{,}00194P_2^2 + 7{,}85P_1 + 310$$

$$F_3(P_3) = 0{,}00482P_3^2 + 7{,}97P_1 + 78$$

Subject to the constraints:

$$150 \leq P_1 \leq 600 \qquad 100 \leq P_2 \leq 400 \qquad 50 \leq P_3 \leq 200$$

The initial values of each generator are:

$$P_{01} = 400\ MW \qquad P_{02} = 300\ MW \qquad P_{03} = 150\ MW$$

The transmission-loss equation is given by:

$$P_L = 0{,}00003P_1^2 + 0{,}00009P_2^2 + 0{,}00012P_3^2$$

where the generated powers and the transmission losses are expressed in MW. The total demand is 850 MW.

10.1 Application of the iterative lambda method

Assume the initial value $\lambda_0 = 8\ \$/MWh$.

We calculate the increment of transmission losses:

$$\frac{\partial P_L}{\partial P_1} = 2(0{,}00003)400 = 0{,}0240$$

$$\frac{\partial P_L}{\partial P_2} = 2(0{,}00009)300 = 0{,}0540$$

$$\frac{\partial P_L}{\partial P_3} = 2(0{,}00012)150 = 0{,}0360$$

The total transmission losses are 15.6 MW.

$$E = \sum_{i=1}^{N} P_i - P_D - P_L = \sum_{i=1}^{3} \frac{\lambda\left(1 - \frac{\partial P_L}{\partial P_i}\right) - b_i}{2c_i} - P_D - P_L$$

$$= \frac{8(1-0{,}024)-7{,}92}{2.0{,}001562} + \frac{8(1-0{,}054)-7{,}85}{2.0{,}00194} + \frac{8(1-0{,}036)-7{,}97}{2.0{,}00482} - 850 - 15{,}6$$

$$= -1000{,}898\ MW$$

$$E' = \sum_{i=1}^{3} \frac{1 - 2B_{ii}P_i}{2c_i} = \frac{8(1-0{,}024)}{2.0{,}001562} + \frac{8(1-0{,}054)}{2.0{,}00194} + \frac{8(1-0{,}036)}{2.0{,}00482} = 656{,}2343\ MW$$

Then:

$$\lambda^1 = \lambda^0 - \frac{E(\lambda^0)}{E'(\lambda^0)} = 8 - \frac{-1000{,}898}{656{,}2343} = 9{,}5252\ \$/MWh$$

Therefore:

$$P_1 = \frac{9{,}5252(1-0{,}024)-7{,}92}{2.0{,}001562} = 440{,}65\ MW$$

$$P_2 = \frac{9{,}5252(1-0{,}054)-7{,}85}{2.0{,}00194} = 299{,}185\ MW$$

$$P_3 = \frac{9{,}5252(1-0{,}036)-7{,}97}{2.0{,}00482} = 125{,}756\ MW$$

We calculate the increment of transmission losses for the second iteration:

$$\frac{\partial P_L}{\partial P_1} = 2(0{,}00003)400{,}65 = 0{,}0264$$

$$\frac{\partial P_L}{\partial P_2} = 2(0{,}00009)299{,}185 = 0{,}0538$$

$$\frac{\partial P_L}{\partial P_3} = 2(0{,}00012)125{,}756 = 0{,}0301$$

The total transmission losses are then 15.78 MW.

$$E = \frac{9{,}5252(1-0{,}0264)-7{,}92}{2.0{,}001562} + \frac{9{,}5252(1-0{,}0538)-7{,}85}{2.0{,}00194} + \frac{9{,}5252(1-0{,}0301)-7{,}97}{2.0{,}00482} - 850 - 15{,}78 = -1{,}1845\ MW$$

$$E' = \frac{9{,}5252(1-0{,}0264)}{2.0{,}001562} + \frac{9{,}5252(1-0{,}0538)}{2.0{,}00194} + \frac{9{,}5252(1-0{,}0301)}{2.0{,}00482} = 656{,}1296\ MW$$

Then:

$$\lambda^2 = 9{,}5252 - \frac{-1{,}1845}{656{,}1296} = 9{,}527\ \$/MWh$$

Therefore:

$$P_1 = \frac{9{,}527(1-0{,}0264)-7{,}92}{2.0{,}001562} = 443{,}91\ MW$$

$$P_2 = \frac{9{,}527(1-0{,}0538)-7{,}85}{2.0{,}00194} = 300{,}11\ MW$$

$$P_3 = \frac{9{,}527(1-0{,}0301)-7{,}97}{2.0{,}00482} = 131{,}767\ MW$$

The following table summarizes the iterative process used to solve this problem.

Iterations	P_1 $[MW]$	P_2 $[MW]$	P_3 $[MW]$	P_L $[MW]$	λ $[\$/MWh]$
0	400	300	150	15.6	9.5252
1	440.65	299.18	125.75	15.78	9.527
2	433.91	300.11	131.76	15.84	9.5285
3	435.87	299.94	130.42	15.83	9.5283
4	435.13	299.99	130.71	15.83	9.5284

10.2 Application of the first-order gradient method

We consider the third unit as the dependent unit. Then:

$$\Delta F = \left(\frac{dF_1}{dP_1} - \frac{dF_3}{dP_3}\right)\Delta P_1 + \left(\frac{dF_2}{dP_2} - \frac{dF_3}{dP_3}\right)\Delta P_2 = (-0{,}2464)\Delta P_1 + (-0{,}4020)\Delta P_2$$

and $F = 8200{,}47\ \$/h$

We wish to decrease F (ΔF negative); therefore, we must increase P_2 (ΔP_2 positive) since its coefficient is negative. Thus, 50 MW must be added to P_2 and 50 MW must be subtracted from P_3.

$$\left.\begin{aligned}P_1 &= 400\ MW\\ P_2 &= 350\ MW\\ P_3 &= 100\ MW\end{aligned}\right\}\text{2nd iteration}$$

We obtain:

$$\Delta F = (0{,}2356)\Delta P_1 + (-0{,}2740)\Delta P_2$$

$$F = 8197{,}27\ \$/h$$

The following table summarizes the iterative process used to solve this problem.

Iterations	$P_1\ [MW]$	$P_2\ [MW]$	$P_3\ [MW]$	$F\ [\$/h]$	$\Delta P_2\ [MW]$	$\Delta P_3\ [MW]$	$\Delta P_3\ [MW]$
1	400	300	150	8200,47	0	+50	-50
2	400	350	100	8197,27	0	-25	+25
3	400	325	125	8194,63	0	+12,5	-12,5
4	400	337,5	112,5	8194,92	-10	0	+10
5	390	337,5	112,5	8194,38	+5	0	-5
6	395	337,5	171,5	8194,48	0	-2,5	+2,5
7	395	335	120	8194,38			

10.3 Application of the second-order gradient method

We calculate the first derivatives of the function F_i:

$$F_1' = 9{,}1696\ \$/MWh \qquad F_1'' = 0{,}003124\ \$/MWh/MW$$

$$F_2' = 9{,}0140\ \$/MWh \qquad F_2'' = 0{,}003880\ \$/MWh/MW$$

$$F_3' = 9{,}4160\ \$/MWh \qquad F_3'' = 0{,}009640\ \$/MWh/MW$$

We use equation (40):

$$\begin{bmatrix} F_1'' + F_3'' & F_3'' \\ F_3'' & F_2'' + F_3'' \end{bmatrix}\begin{bmatrix}\Delta P_1\\ \Delta P_2\end{bmatrix} = -\begin{bmatrix} F_1' - F_3' \\ F_2' - F_3' \end{bmatrix}$$

$$\begin{bmatrix} 0{,}012764 & 0{,}00964 \\ 0{,}009640 & 0{,}01352 \end{bmatrix}\begin{bmatrix}\Delta P_1\\ \Delta P_2\end{bmatrix} = -\begin{bmatrix} -0{,}2464 \\ -0{,}4020 \end{bmatrix}$$

The solution is:

$$\begin{bmatrix}\Delta P_1\\ \Delta P_2\end{bmatrix} = \begin{bmatrix} -6{,}8301 \\ 34{,}6030 \end{bmatrix}$$

Therefore:

$$P_1 = 400 - 6{,}8301 = 393{,}17\ MW$$

$$P_2 = 300 + 34{,}603 = 334{,}60\ MW$$

$$P_3 = 850 - P_1 - P_2 = 122{,}23\ MW$$