REVIEW: FUNDAMENTALS OF AC SYSTEMS

1 Introduction

The development, and especially the interconnection, since the end of the 19th century, of the networks for the production, transmission and distribution of electrical energy have forced electricity companies to make joint choices on the nature of the electrical system; direct or alternating voltages, single-phase or three-phase, frequency values, etc. These choices have been dictated by technical considerations, but also economic ones. They have conditioned the evolution of energy networks and have frozen over the decades some fundamental parameters which no longer necessarily correspond to an optimum today. This is how the voltage was chosen to be alternating, in order to be able to exploit the properties of the transformer to transport energy at high voltage (to minimize losses) and use it at low voltage (for safety reasons). The three-phase system also offers advantages over single-phase: simpler creation of a rotating field (synchronous and asynchronous machines), possible use of different transformer couplings, use of phase-to-neutral or phase-to-phase voltages as needed, constant instantaneous power, etc. As for frequency, it was chosen based on technical constraints relating to network equipment and rotating machines. The optimal frequency today would certainly be higher than 50 or 60 Hz.

Despite the development of power electronics allowing the use of direct current connections, the study of electrical networks necessarily involves the study of three-phase systems.

2 Representation of sinusoidal signals

The oscillation of voltage and current in an ac system is modelled by a sinusoidal curve, meaning that it is mathematically described by the trigonometric functions of sine or cosine. In these functions, time appears not in the accustomed units of seconds or minutes, but in terms of an angle.

A sinusoidal function is specified by three parameters: amplitude, frequency, and phase. Plotted against angle on the horizontal axis, the height of the sine curve is simply the value of the sine for each angle, scaled up by a factor corresponding to the amplitude. As the angle is increased, it eventually describes a complete circle, and the function repeats itself. In the context of sinusoidal functions, angles are often specified in units of radians (rad) rather than degrees.

2.1 Sinusoids

Consider the sinusoidal signal $s(t)$:

$$s(t) = A_m cos(\omega t + \varphi) \tag{1.1}$$

where

s is the instantaneous value of the signal;

A_m is the amplitude or the maximum value;

$\omega = 2\pi f$ in rad/s is the angular frequency and f the frequency in Hz

$\theta = \omega t + \varphi$ is the argument in rad;

φ is the phase in rad;

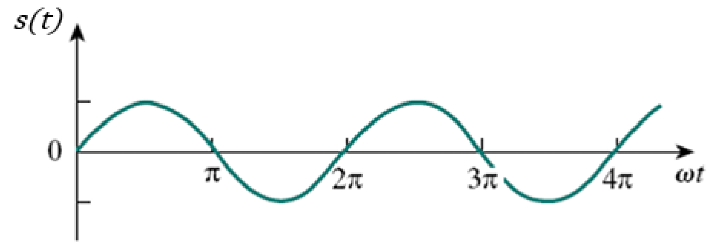
The sinusoid repeats itself every T seconds; thus, T is called the period of the sinusoid.

$T = \frac{2\pi}{\omega} = \frac{1}{f}$ in seconds is the period of the signal;

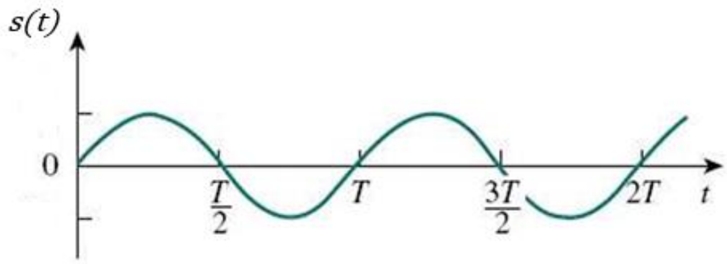
That is, s has the same value at $t + T$ as it does at t and $s(t)$ is said to be periodic.

$$s(t + T) = s(t) \tag{1.2}$$

The sinusoid is shown in Fig. 1.1a as a function of its argument and in Fig. 1.1b as a function of time.



(a)



(b)

Fig. 1.1: Sinusoidal signal

The mean or average value:

$$A_{av} = \frac{1}{T}\int_0^T s(t)dt = 0 \tag{1.3}$$

The effective or RMS value:

$$A_{eff} = \sqrt{\frac{1}{T}\int_0^T s^2(t)dt} = \frac{A_m}{\sqrt{2}} \tag{1.4}$$

2.2 Phasors

A sine wave or signal is entirely characterized by three variables A_m, ω and φ. Another quantity can be associated with it, a vector, also characterized by the same three variables. This vector $\vec{s}(t)$ has amplitude A_m and rotates in the counter clockwise direction at angular velocity ω. At time $t = 0$, the angle between the vector and the reference axis is φ.

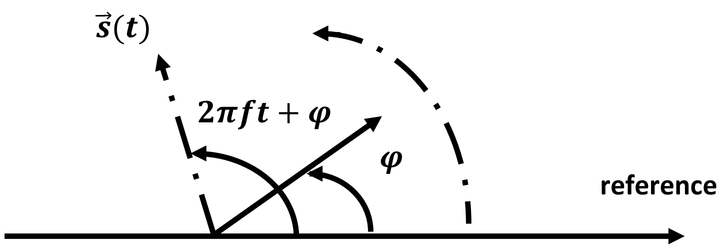


Fig. 1.2 : Fresnel vector mapping

If we consider two sinusoidal signals $s_1(t)$ and $s_2\ (t)$ of different frequencies *f₁* and *f₂*, the two associated vectors $\vec{s}_1(t)$ and $\vec{s}_1(t)$ will rotate at different speeds. Their phase shift will therefore vary over time. On the other hand, if all the signals considered are at a fixed and known frequency *f*, this information is no longer of interest; all the vectors associated with the signals rotate at the same speed: they are fixed relative to each other. Under these conditions, only the representation of the vectors at time $t = 0$ is of interest, they are then called phasors. On the diagram, we can read amplitudes of the signals and their phases. A phasor is a shorthand way to characterize a sine wave, specifying its magnitude and angle (in relation to a reference).

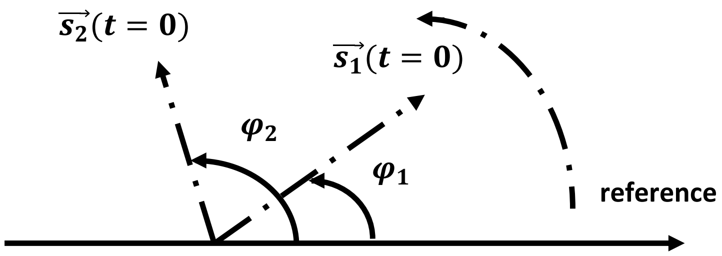


Fig. 1.3 : Phasors

2.3 Complex notation

Each vector $\vec{s}(t)$ representing a sine wave $s(t)$ can also be associated with a complex number, denoted $\bar{s}(t)$, whose real part is the projection of $\vec{s}(t)$ on the reference axis, and the imaginary part the projection on a quadrature axis (Fig. 1.4).
The signal $s(t)$ defined by equation (1.1) can be expressed as the real part of $\bar{s}(t)$ as follows:

$$s(t) = \Re(A_m e^{j\omega t} e^{\varphi}) \qquad (1.5)$$

Taking the frequency as a given, the main aspect of timing that has practical significance is the phase shift. Then, we can associate the signal $s(t)$ with a simpler complex number, involving only the amplitude and phase of $s(t)$:

$$s(t) = A_m cos(\omega t + \varphi) \quad \rightarrow \quad \bar{s} = A_m e^{j\varphi} \qquad (1.6)$$

In electrical engineering, we prefer the polar notation $\bar{s} = A_m \angle \varphi$

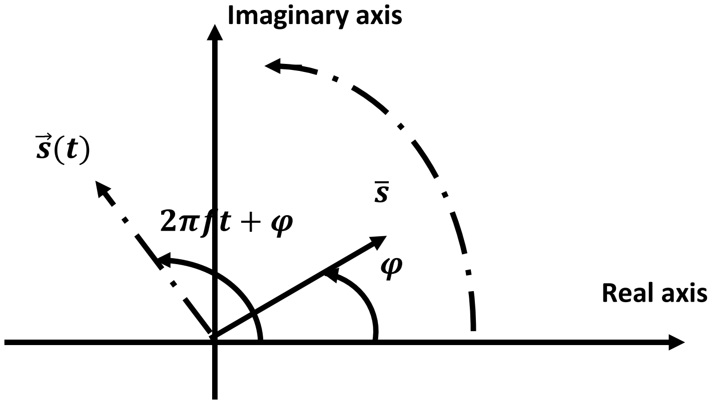


Fig. 1.4 : Complex Representation

Important notes:

RMS value and peak value: In practical situations when dealing with ac circuits, we are interested in average as opposed to instantaneous values of current, voltage, and power. In other words, we want to know what happens over the course of many cycles, not within a single cycle. Thus, we describe current and voltage in terms of rms values. For a purely sinusoidal signal $s(t)$, the rms value A_{eff} and the peak value A_m are related by the following relation: $A_m = \sqrt{2}A_{eff}$. Therefore, we will use the convention that states the rms value as opposed to the full wave amplitude: $\bar{s} = A_{eff}e^{j\varphi}$.

Phasor representation: To get the phasor corresponding to a sinusoid, we first express the sinusoid in the cosine form so that the sinusoid can be written as the real part of a complex number. Then we take out the time factor, and whatever is left is the phasor corresponding to the sinusoid. By suppressing the time factor, we transform the sinusoid from the time domain to the phasor domain.

This transformation is summarized as follows:

Time domain representation	Phasor domain representation
$s(t) = A_m cos(\omega t + \varphi)$	$\bar{s} = Ae^{j\varphi} = A\angle\varphi$
$s(t) = A_m sin(\omega t + \varphi)$	$\bar{s} = A\angle\varphi - 90$

Basics of complex calculation

A complex number z can be written in different forms as:

Rectangular form: $z = x + jy$

Exponential form: $z = re^{j\varphi}$

Polar form: $z = r\angle\varphi$

The idea of phasor representation is based on Euler's identity: $e^{j\varphi} = cos\varphi + jsin\varphi$

If $\bar{A} = a + jb = A\angle\theta$ then

$$a = A.cos\theta \quad b = A.sin\theta \quad A = \sqrt{a^2 + b^2} \qquad \theta = arctg\left(\frac{b}{a}\right)$$

If $\bar{A} = a + jb = A\angle\theta_a$ and $\bar{B} = c + jd = B\angle\theta_b$ then

$$\bar{A} \pm \bar{B} = (a \pm c) + j(b \pm d)$$

$$\bar{A}.\bar{B} = AB\angle(\theta_a + \theta_b) \qquad \frac{\bar{A}}{\bar{B}} = \frac{A}{B}\angle(\theta_a - \theta_b)$$

$$\bar{A}^2 = A^2\angle 2\theta_a \qquad \sqrt{\bar{A}} = \pm\sqrt{A}\angle\theta_a/2$$

The conjugate of $\bar{A}$ is : $\bar{A}^* = a - jb = A\angle{-\theta_a}$

3 Phasor relationship for circuit elements

Now that we know how to represent a voltage or current in the phasor or frequency domain, one may legitimately ask how we apply this to circuits involving the passive elements R, L, and C. What we need to do is to transform the voltage-current relationship from the time domain to the frequency domain for each element.

Consider a linear dipole supplied in steady state with a sinusoidal voltage $v(t)$ and carrying a current $i(t)$.

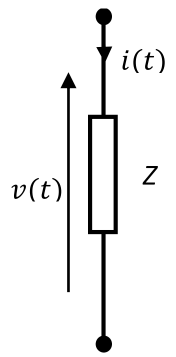


Fig. 1.5 : Linear dipole of impedance Z

3.1 Resistor

If the current through a resistor R is

$$i(t) = I_m \cos(\omega t + \theta_i)$$

the voltage across it is given by Ohm's law as

$$v(t) = R.i(t) = R.I_m \cos(\omega t + \theta_i)$$

The phasor form of this voltage is

$$\bar{V} = R.I \angle\theta_i \qquad (1.7)$$

But the phasor representation of the current is $\bar{I} = I \angle\theta_i$

Hence,

$$\bar{V} = R.\bar{I} \qquad (1.8)$$

Current and voltage are in phase; there is no phase shift (Fig. 1.6).

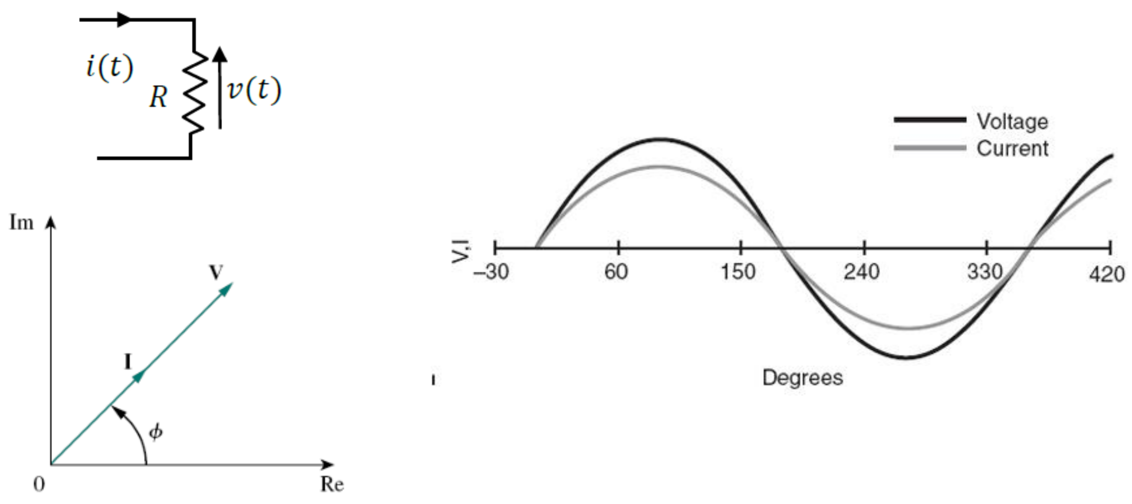


Fig. 1.6 : AC voltage and current across a resistor

3.2 Inductor

For the inductor L, assume the current through it is

$$i(t) = I_m \cos(\omega t + \theta_i)$$

The voltage across the inductor is

$$v(t) = L\frac{di(t)}{dt} = -L\omega I_m \sin(\omega t + \theta_i) = L\omega I_m \cos\left(\omega t + \theta_i + \frac{\pi}{2}\right)$$

Thus, with $\bar{I} = I\,\angle\theta_i$

$$\bar{V} = L\omega.I\ \angle(\theta_i + 90°) = L\omega.I\ \angle\theta_i\ \angle 90° = L\omega\bar{I}\ \angle 90° = jL\omega\bar{I} \tag{1.9}$$

The voltage and current are 90° out of phase. Specifically, the current lags the voltage by 90°. Fig. 1.7 shows the voltage-current relations for the inductor and the phasor diagram.

$X = L\omega$ is called inductive reactance of the circuit.

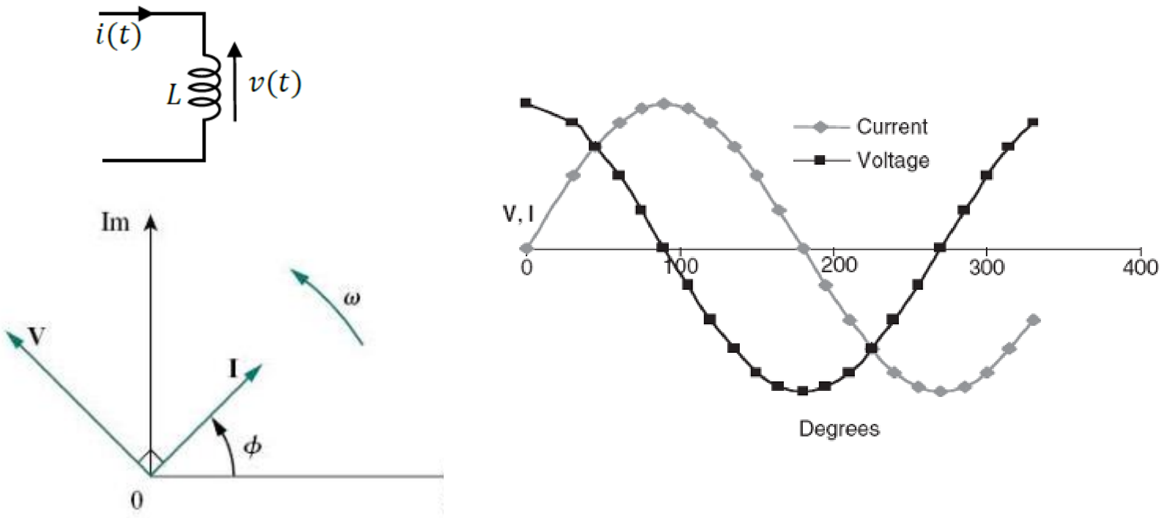


Fig. 1.7 : AC voltage and current across an inductor

3.3 Capacitor

For the capacitor C, assume the current through it is

$$i(t) = I_m \cos(\omega t + \theta_i) = C\frac{dv(t)}{dt}$$

the voltage across the capacitor is then

$$v(t) = \frac{1}{C}\int i(t)\,dt = \frac{1}{C\omega} I_m \sin(\omega t + \theta_i) = \frac{1}{C\omega} I_m \cos(\omega t + \theta_i - \frac{\pi}{2})$$

By following the same steps as we took for the inductor we obtain:

$$\bar{V} = \frac{1}{C\omega}.I\ \angle(\theta_i - 90°) = \frac{1}{C\omega}.I\ \angle\theta_i\ \angle(-90°) = \frac{1}{C\omega}\bar{I}\angle(-90°) = \frac{1}{jC\omega}\bar{I} \tag{1.10}$$

showing that the current and voltage are 90° out of phase. To be specific, the current leads the voltage by 90°. Fig. 1.8 shows the voltage-current relations for the capacitor and gives the phasor diagram.

$X = \dfrac{1}{C\omega}$ is the capacitive reactance.

Following is the summary of the time-domain and phasor-domain representations of the circuit elements.

Element	Time domain	Frequency domain
R	$v = R.i$	$\bar{V} = R.\bar{I}$
L	$v = L\dfrac{di}{dt}$	$\bar{V} = jL\omega\bar{I}$
C	$i = C\dfrac{dv}{dt}$	$\bar{V} = \dfrac{1}{jC\omega}\bar{I}$

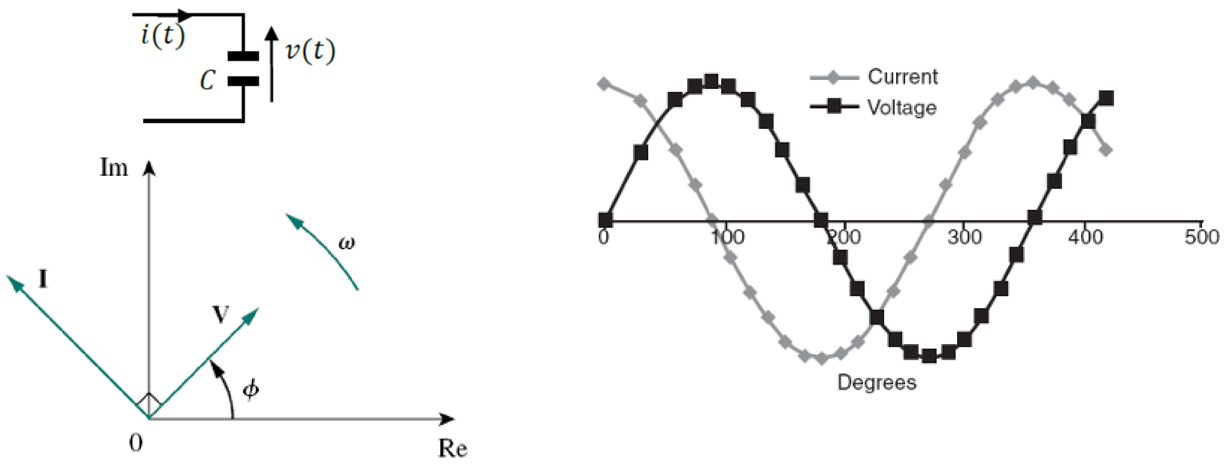


Fig. 1.8 : AC voltage and current across a capacitor

3.4 Impedance and admittance

In the preceding section, we obtained the voltage-current relations for the three passive elements as

$$\bar{V} = R.\bar{I} \quad \bar{V} = jL\omega\bar{I} \qquad \bar{V} = \frac{1}{jC\omega}\bar{I}$$

These equations may be written in terms of the ratio of the phasor voltage to the phasor current as

$$\frac{\bar{V}}{\bar{I}} = R \qquad \frac{\bar{V}}{\bar{I}} = jL\omega \qquad \frac{\bar{V}}{\bar{I}} = \frac{1}{jC\omega}$$

From these thee expressions, we obtain Ohm's law in phasor form for any type of element as

$$Z = \frac{\bar{V}}{\bar{I}} \quad \text{or} \quad \bar{V} = Z.\bar{I} \tag{1.11}$$

where Z is a frequency-dependent quantity known as impedance, measured in ohms.

The impedance Z of a circuit is the ratio of the phasor voltage V to the phasor current I, measured in ohms (Ω). The impedance represents the opposition which the circuit exhibits to the flow of sinusoidal current. Although the impedance is the ratio of two phasors, it is not a phasor, because it does not correspond to a sinusoidally varying quantity.

The admittance Y is the reciprocal of impedance, measured in siemens (S). The admittance Y of an element (or a circuit) is the ratio of the phasor current through it to the phasor voltage across it, or

$$Y = \frac{\bar{I}}{\bar{V}} \quad \text{or} \quad Y = \frac{1}{Z} \tag{1.12}$$

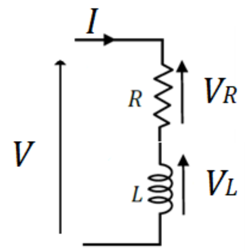
As a complex quantity, the impedance and admittance may be expressed in rectangular form as

$$Z = R + jX \tag{1.13}$$

$$Y = G + jB \tag{1.14}$$

Where R: is the resistance; X: is the reactance; G: is the conductance; B: is the susceptance.

3.5 Example



$R = 6\Omega \quad L\omega = 8\Omega$

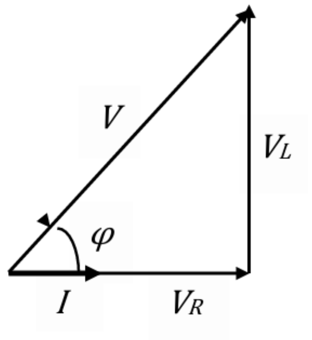


Fig. 1.9: RL Circuit

Determine the phase shift between V and I; and calculate V for $I = 1A$.

According to the mesh law:

$$\bar{V} = \overline{V_R} + \overline{V_L} = R.\bar{I} + jL\omega.\bar{I} = (R + jL\omega).\bar{I} \tag{1.15}$$

or

$$\bar{V} = \bar{Z}.\bar{I} \quad \text{with} \quad \bar{Z} = R + jL\omega \tag{1.16}$$

The current is lagging the voltage by an angle φ such that : $tg\varphi = L\omega/R$.

$$tg\varphi = \frac{L\omega}{R} = \frac{8}{6} = 1{,}33 \qquad \text{thus} \qquad \varphi = 59°$$

$$Z = \sqrt{6^2 + 8^2} = 10\Omega \qquad \text{thus} \qquad V = Z.I = 10 \times 1 = 10V$$

Note: Series/Parallel connections

The analysis using laws of meshes, nodes, and impedance connections is identical to that of dc circuits with resistors.

If Z_1 and Z_2 are connected in series, then $\bar{Z} = \overline{Z_1} + \overline{Z_2}$

If Z_1 and Z_2 are connected in parallel, then $\frac{1}{\bar{Z}} = \frac{1}{\overline{Z_1}} + \frac{1}{\overline{Z_2}}$

4 Power in single phase ac networks

The electric power systems specialist is in many instances more concerned with electric power in the circuit rather than the currents. As the power into an element is basically the product of voltage across and current through it, it seems reasonable to swap the current for power without losing any information in describing the phenomenon.

Power is the time rate of expending or absorbing energy, measured in watts (W):

$$p = \frac{dw}{dt} \tag{1.17}$$

Where p is power in watts (W), w is energy in joules (J), and t is time in seconds (s).

From voltage and current equations, it follows that:

$$p = \frac{dw}{dt} = \frac{dw}{dq} \cdot \frac{dq}{dt} = vi \tag{1.18}$$

The power in this equation is a time-varying quantity and is called the instantaneous power. Thus, the power absorbed or supplied by an element is the product of the voltage across the element and the current through it.

It is important to realize that, just like voltage, power is a signed quantity, and that it is necessary to make a distinction between positive and negative power. The electrical engineering community uniformly adopts the passive sign convention, which simply states that the power dissipated by a load is a positive quantity (or, conversely, that the power generated by a source is a positive quantity).

To illustrate the concepts of power, Consider the impedance element of Fig 1.5. We will use a cosine representation of the waveforms of voltage $v(t)$ and instantaneous current in the circuit:

$$i(t) = I_m cos(\omega t + \theta_i) \tag{1.19}$$

$$v(t) = V_m cos(\omega t + \theta_v) \tag{1.20}$$

The instantaneous power dissipated in the circuit is:

$$p(t) = v(t).i(t) = V_m I_m cos(\omega t + \theta_v).cos(\omega t + \theta_i) \tag{1.21}$$

Using the trigonometric identity: $cosA.cosB = \frac{1}{2}[cos(A-B) + cos(A+B)]$

$$p(t) = \frac{1}{2} V_m I_m [cos(\theta_v - \theta_i) + cos(2\omega t + \theta_v + \theta_i)]$$

$$p(t) = \frac{1}{2} V_m I_m [cos(\theta_v - \theta_i) + cos(2(\omega t + \theta_v) - (\theta_v - \theta_i)]$$

$$p(t) = \frac{1}{2} V_m I_m [cos(\theta_v - \theta_i) + cos(2(\omega t + \theta_v)).cos(\theta_v - \theta_i) + sin(2(\omega t + \theta_v)).sin(\theta_v - \theta_i)]$$

$$p(t) = \frac{1}{2} V_m I_m \left[cos(\theta_v - \theta_i).\left(1 + cos2(\omega t + \theta_v)\right) + sin(\theta_v - \theta_i).sin2(\omega t + \theta_v)\right] \tag{1.22}$$

Noting :

$$V = \frac{V_m}{\sqrt{2}} \quad I = \frac{I_m}{\sqrt{2}} \quad \varphi = \theta_v - \theta_i$$

$$p(t) = \underbrace{VIcos\varphi[1 + cos(2\omega t + 2\theta_v)]}_{P_R(t)} + \underbrace{VIsin\varphi[sin(2\omega t + 2\theta_v)]}_{P_X(t)} \tag{1.23}$$

V and I are known as effective values of voltage and current.

The equation shows that the instantaneous power has two parts. The first part $P_R(t)$ is constant or time independent. Its value depends on the phase difference between the voltage and the current. The second part $P_X(t)$ is a sinusoidal function whose frequency is 2ω, which is twice the angular frequency of the voltage or current. The instantaneous power is sketched in Fig. 1.10.

The instantaneous power changes with time and is therefore difficult to measure. The average power is more convenient to measure. It is given by:

$$P = \frac{1}{T}\int_0^T p(t)dt = VIcos\varphi \tag{1.24}$$

The average power corresponds to the power actually transmitted or consumed by the load. It is also called real power, active power or true power, and is measured in watts. This power is the product of the effective values of terminal voltage and current and the cosine of the phase angle, which is, called the power factor (PF). This applies to sinusoidal voltages and currents only.

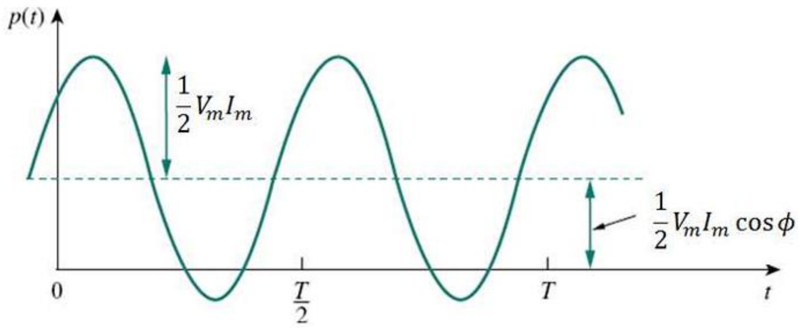


Fig. 1.10: The instantaneous power $p(t)$ entering the circuit

$P_X(t)$ is the energy that is periodically stored in and discharged from the reactance. This stored energy, being shuttled to and from the magnetic field of an inductance or the electric field of a capacitance, adds to the current in the circuit but does not add to the average power. The power that supplies the stored energy in reactive elements is called reactive power and designed by Q.

$$Q = VIsin\varphi \quad (1.25)$$

To emphasize that the Q represents the nonactive power, it is measured in reactive voltampere units (VAr).

In the special case where there is only resistance and no phase shift, we have $\varphi = 0$ and $cos\varphi = 1$, so there is no need to write down the $cos\varphi$. In another special case where the load is purely reactive (having no resistance at all), the phase shift would be $\varphi = 90°$ and $cos\varphi = 0$,, meaning that power only oscillates back and forth, but is not dissipated (the average power is zero). A resistive load (R) absorbs power at all times, while a reactive load (L or C) absorbs zero average power.
If the dipole is inductive ($L\omega > 1/C\omega$), then $Q > 0$; we say that the dipole consumes reactive power.
If the dipole is capacitive ($L\omega < 1/C\omega$), then $Q < 0$; we say that the dipole supplies reactive power.

It is necessary to minimize reactive power flows on an energy network. Indeed, for the same useful (active) power, greater reactive power leads to a greater absorbed current, and consequently, to more losses. This will in fact amount to limiting the phase shift between voltage and current (i.e. having a power factor as close as possible to 1). Where the network is too inductive, it will be appropriate, for example, to add capacitive elements (reactive power compensation by capacitor banks).

Complex Power

We define a quantity called the complex power, designated $\bar{S}$, of which P and Q are components. By definition,

$$\bar{S} = P + jQ = VIcos\varphi + jVIsin\varphi \quad (1.26)$$

It is clear that an equivalent definition of this complex power is:

$$\bar{S} = VIe^{j\varphi} = \bar{V}.\bar{I}^* \quad (1.27)$$

We can represent power as a vector in the complex plane: namely, an arrow of length S (apparent power) that makes an angle with the real axis. This is shown in Fig. 1.11. The angle is the same as the phase difference between voltage and current.

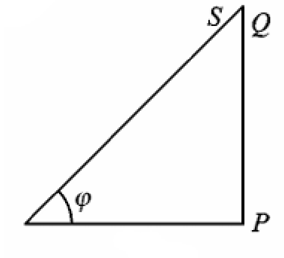
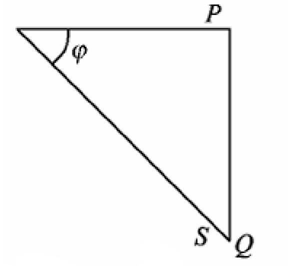
	
a) Inductive load	b) Capacitive load

Fig. 1.11: Power triangle

The apparent power measured in voltampere units (VA) is then:

$$S = V.I = \sqrt{P^2 + Q^2} \tag{1.28}$$

The power factor is expressed by: $$cos\varphi = \frac{P}{S} \tag{1.29}$$

If the load impedance is $\bar{Z} = R + jX$ then $\bar{V} = \bar{Z}.\bar{I}$, thus:

$$\bar{S} = \bar{V}.\bar{I}^* = \bar{Z}.\bar{I}.\bar{I}^* = \bar{Z}.I^2 = R.I^2 + jXI^2$$

Therefore

$$P = RI^2 \text{ and } Q = XI^2 \tag{1.30}$$

In another way, we can write:

$$\bar{S} = \bar{V}.\bar{I}^* = \bar{V}.\left(\frac{\bar{V}}{\bar{Z}}\right)^* = \bar{V}.\frac{\bar{V}^*}{\bar{Z}^*} = \frac{V^2}{\bar{Z}^*} \quad \rightarrow \quad \bar{Z} = \frac{V^2}{\bar{S}^*}$$

It follows that for a pure resistance, $Q = 0$ and $P = V^2/R$ whereas for an inductance (or a capacitance) $P = 0$ and $Q = V^2/X$.

Conservation of Power

When working with complex power we often use the Theorem of the conservation of power. This indicates that the total active power delivery of all sources is equal to the sum of all the active powers dissipated in the various components, and the total reactive power is the sum of the reactive powers dissipated in the various components. This is valid in a network consisting of multiple sources and consumers, each of them independent of each other. The theorem applies to three-phase systems. Thereby one assumes all currents and voltages to be purely sinusoidal and having the same frequency.

Reactive power and power factor correction

A low power factor is undesirable for utilities in terms of operating efficiency and economics. Most customers, especially small customers, are only charged for the real power they consume. At the same time, the presence of reactive power oscillating through the lines and equipment is associated with additional current. While reactive power as such is not consumed, it nonetheless causes the utility to incur costs, both in the form of additional losses and in the form of greater capacity requirements.

Owing to its property of occupying lines and equipment while doing no useful work, reactive power has been referred to jokingly as the cholesterol of power lines.
From the expression of the active power; it can be seen that the apparent power will be larger than *P* if the power factor is less than 1. Thus the current *I* that must be supplied will be larger for $PF < 1$, than it would be for $PF = 1$, even though the average power supplied is the same in either case. A larger current cannot be supplied without additional cost to the utility company. Thus, it is in the power company's (and its customer's) best interest that major loads on the system have power factor as close to 1 as possible.

In order to maintain the power factor close unity, power companies install bank of capacitors thought the network as needed. They also impose an additional charge to industrial consumers who operate at low power factors. Since industrial loads are inductive and have low lagging power factors, it is beneficial to install capacitors to improve the power factor. Preferably, this compensation is placed near the load, so as to minimize the distance that reactive power must travel through the lines. This consideration is not important for residential and small commercial customers because their power factors are close to unity.
Consider the power triangle in Fig; 1.12. If the original inductive load has apparent power S_1, then

$$P = S_1 cos\varphi_1$$

$$Q_1 = S_1 sin\varphi_1 = Ptg\varphi_1$$

If we desire to increase the power factor from $cos\varphi_1$ to $cos\varphi_2$ without altering the real power (i.e., $P = S_2 cos\varphi_2$), then the new reactive power is

$$Q_2 = Ptg\varphi_2$$

The reduction in the reactive power is caused by the shunt capacitor, that is,

$$Q_C = Q_1 - Q_2 = P(tg\varphi_1 - tg\varphi_2) \tag{1.31}$$

Thus,

$$Q_C = \frac{V^2}{X_C} = \omega C V^2$$

The value of the required shunt capacitance C is determined as

$$C = \frac{Q_C}{\omega V^2} = \frac{P(tg\varphi_1 - tg\varphi_2)}{\omega V^2} \tag{1.32}$$

Note that the real power *P* dissipated by the load is not affected by the power factor correction because the average power due to the capacitance is zero.

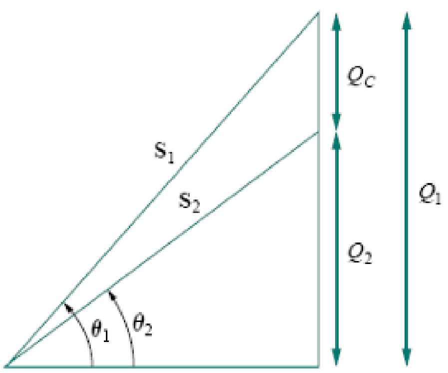


Fig. 1.12: Power triangle illustrating power factor correction

5 Three-phase systems

The major portion of all electric power presently used in generation, transmission, and distribution uses balanced three-phase systems. Three-phase operation makes more efficient use of generator copper and iron. A three-phase line experiences half Joule losses to provide the same power P than a single-phase line.

A symmetric three-phase system is supplied by three voltages with equal amplitudes, having equal frequency, but each phase shifted by ±120° (2π/3) with respect to the others.

If the generated voltages reach their peak values in the sequential order 123 (or abc), the generator is said to have a positive phase sequence. If the order phase is 132 (acb), the generator is said to have a negative phase sequence. Thus, the phase sequence is the time order in which the voltages pass through their respective maximum values.

5.1 Phase voltages

The power distribution network uses three phases and a neutral. The voltages v_1, v_2 and v_3 taken between phase and neutral, i.e. with respect to a common point, are called phase-to-neutral voltages or phase voltages.

Fig. 1.13 shows the 123 sequence or positive sequence. In this phase sequence, V_1 leads V_2, which in turn leads V_3. This sequence is produced when the rotor rotates counter clockwise.

Fig. 1.13 : Phase voltages

Mathematically we can write the three voltages

$$v_1(t) = V\sqrt{2}.cos(\omega t)$$

$$v_2(t) = V\sqrt{2}.cos(\omega t - 2\pi/3) \qquad (1.23)$$

$$v_3(t) = V\sqrt{2}.cos(\omega t - 4\pi/3)$$

In phasor representation we can write

$$\overline{V_1} = V\angle 0°$$

$$\overline{V_2} = V\angle -120° \qquad (1.24)$$

$$\overline{V_3} = V\angle -240° = V\angle 120°$$

Fig. 1.14 shows the time variation of these three voltages.

The sum of three sinusoidal quantities forming a balanced system is zero. We can check that:

$$v_1 + v_2 + v_3 = 0.$$

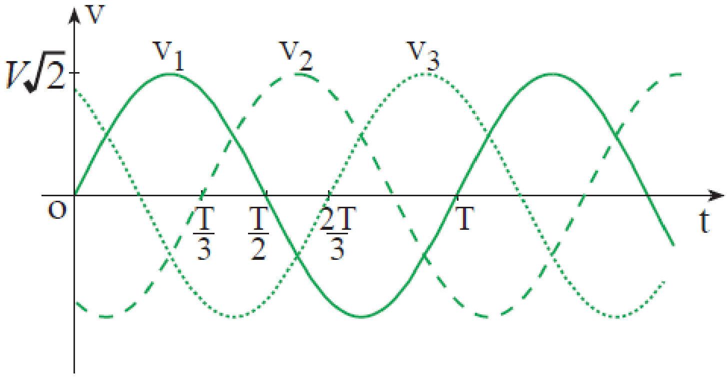


Fig. 1.14: Time variation of the voltages of the phases

Phase shift operator a

The rotation operator a, is the cube root of unity:

$$a = a\,\angle 120° = e^{j2\pi/3} = \cos\left(\frac{2\pi}{3}\right) + j\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + j.\frac{\sqrt{3}}{2} \tag{1.25}$$

Multiplying a complex by a gives a complex with the same amplitude phase-shifted by 120° (or $2\pi/3$). Hence:

$$\overline{V_2} = a^2\overline{V_1} \quad \text{et} \quad \overline{V_3} = a\,\overline{V_1}\,.$$

This result is consistent with the expression $1 + a + a^2 = 0$ given by mathematical calculation.

5.2 Phase to phase voltages

Phase to phase or line to line or simply line voltage is measured between two phases. In the three-phase system three different phase to phase voltages occur between the phases u_{12} , u_{23} and u_{31} (Fig. 1.15). These three voltages are phase shifted by 120°, analogous to the phase voltages.

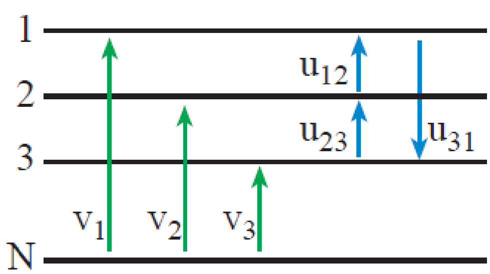


Fig. 1.15 : Phase and phase to phase voltages

Hence,

$$\begin{aligned} u_{12}(t) &= v_1 - v_2 = U\sqrt{2}.cos(\omega t + \pi/6) \\ u_{23}(t) &= v_2 - v_3 = U\sqrt{2}.cos(\omega t - \pi/2) \\ u_{31}(t) &= v_3 - v_1 = U\sqrt{2}.cos(\omega t + 5\pi/6) \end{aligned} \tag{1.26}$$

and in phasor representation:

$$\overline{U_{12}} = \overline{V_1} - \overline{V_2} = (1-a^2)\overline{V_1} = \sqrt{3}V \angle 30° = U \angle 30°$$
$$\overline{U_{23}} = \overline{V_2} - \overline{V_3} = (a^2-a)\overline{V_1} = \sqrt{3}V\angle -90° = U \angle -90° \quad (1.27)$$
$$\overline{U_{31}} = \overline{V_3} - \overline{V_1} = (a-1)\overline{V_1} = \sqrt{3}V\angle 150° = U \angle 150°$$

Fig. 1.16 shows these phasors.

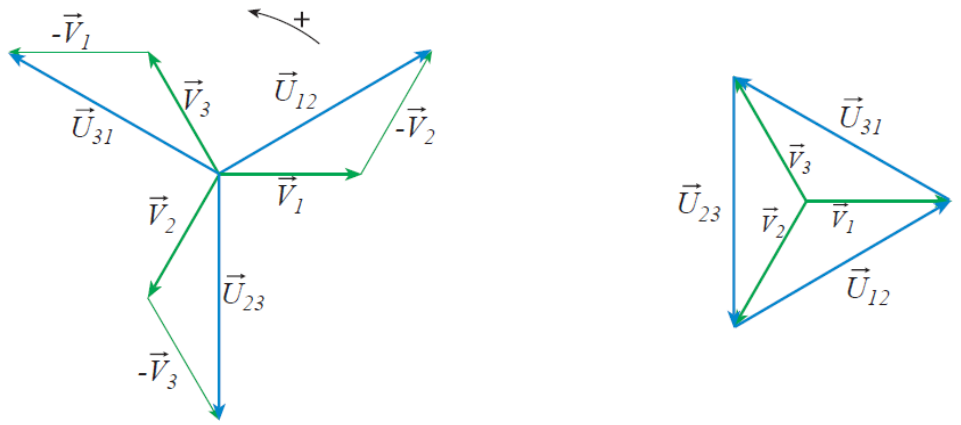


Fig. 1.16 : phasor diagram showing phase and line voltages

From Fig. 1.17, we draw:

$$U = 2V.cos(30°) = V\sqrt{3}$$

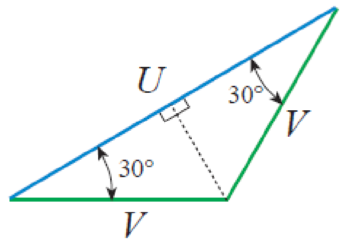


Fig. 1.17 : Phase and line voltages relationship

If the network is balanced, the system formed by the three line voltages is balanced and of positive sequence (we check on the diagram or by calculation that $u_{12} + u_{23} + u_{31} = 0$). It is a balanced three-phase system leading by 30° the phase voltages system.

Fig. 1.18 shows the time variation of these three voltages.

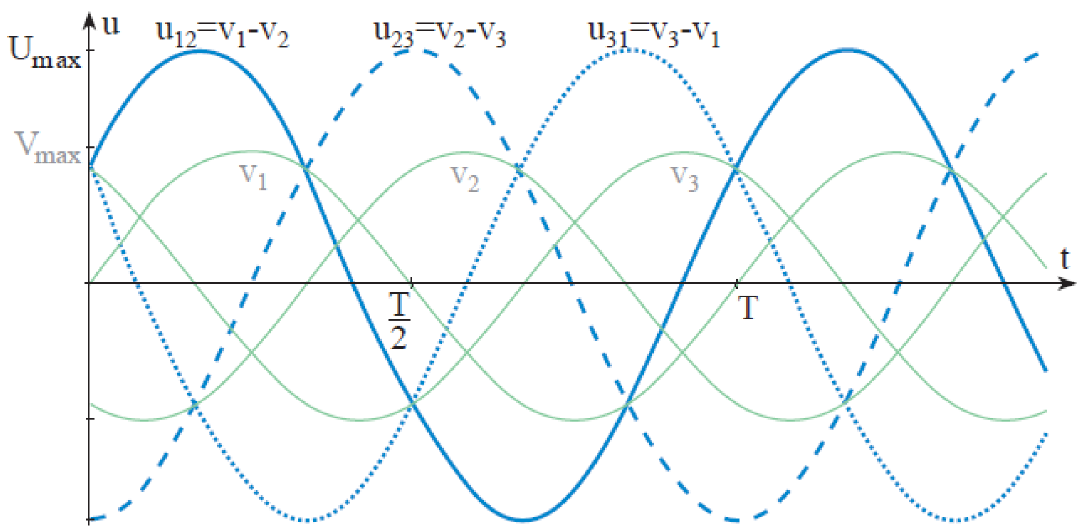


Fig. 1.18: Time variation of phase and line voltages

5.3 Connection modes

A balanced three-phase source can be obtained by connecting three sinusoidal voltage sources with the same amplitude and frequency and 120° phase shifting. If we consider these single-phase sources as independent, two problems arise: no common voltage reference and a 6-wire system, therefore no advantage. It is therefore necessary to connect certain wires. There are two types of connection: star (Y) and delta (Δ) connection. Star connection provides the voltage reference, which is called the neutral (N). Delta connection has neither neutral nor phase-to-neutral voltages. However, it does present two types of current: line currents and phase currents.
Three-phase systems generally have loads distributed across the three phases. As with generators, it is possible to connect these loads in a star or delta configuration. The method of connecting loads allows for the presentation of phase-to-neutral or phase-to-phase voltage values to the loads. Two three-phase loads are considered equivalent if their power consumption is identical. For each load system, it is possible to determine the equivalent star or delta system.

1) Y-Y Connection

In this section, generators and loads are considered to be in balanced operation. The three voltage generators are connected in a star configuration with a common point N and deliver three balanced sinusoidal voltages $v_1,\ v_2$ and v_3. They are connected to three identical load impedances Z connected in a star configuration around the common point N', via three lines numbered 1, 2, 3 and a neutral line between N and N' (Fig. 1.19).

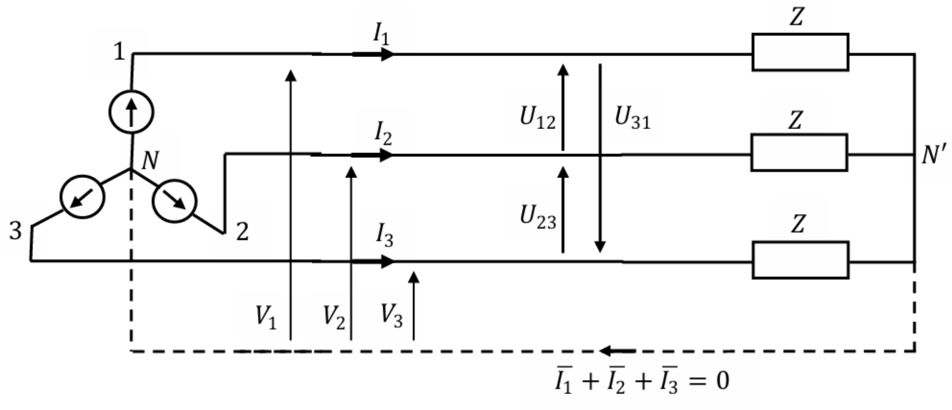


Fig. 1.19 : Y-Y connection

On note $i_1,\ i_2$ et i_3 les trois courants de ligne ; ces courants traversent les trois impédances de charge. We denote by $i_1,\ i_2$ and i_3 the three line currents; these currents flow through the three load impedances. In balanced operation, the current in the neutral is zero ($i_1 + i_2 + i_3 = 0$), so it can be removed without changing the operation of the circuit (dotted line drawing). N and N' are therefore at the same potential, whether a neutral is present or not.

To simplify the study of three-phase circuits (regardless of the chosen coupling mode), we will try to reduce the design to an equivalent single-phase diagram. Studying a single phase is sufficient, as the behavior of the other two is identical to within 120° or 240°.

Since N and N' are at the same potential, we can write:

$$\overline{V_1} = \bar{Z}.\overline{I_1}$$
$$\overline{V_2} = \bar{Z}.\overline{I_2} \qquad (1.28)$$
$$\overline{V_3} = \bar{Z}.\overline{I_3}$$

Which leads to an extremely simple equivalent single-phase diagram (Fig. 1.20):

$$\bar{V} = \bar{Z}.\bar{I} \qquad (1.29)$$

This relationship applies to all three phases up to phase shifts.

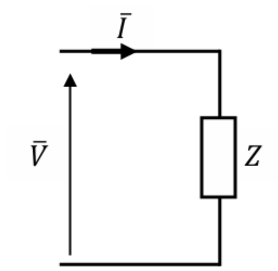


Fig. 1.20 : Equivalent single-phase diagram (Y-Y case)

2) Y-Δ connection

For this connection, the generators are connected in a star configuration and the load impedances in a delta configuration (Fig. 1.21).

Using a delta load does not allow for a neutral. The phase currents j_{12}, j_{23} and j_{31} flowing through the load impedances are called phase currents. The relationship between the phase and line currents can be obtained by:

$$\overline{I_1} = \overline{J_{12}} - \overline{J_{31}} = J\sqrt{3}\angle -30° = I\angle -30°$$
$$\overline{I_2} = \overline{J_{23}} - \overline{J_{12}} = J\sqrt{3}\angle -150° = I\angle -150° \qquad (1.30)$$
$$\overline{I_3} = \overline{J_{31}} - \overline{J_{23}} = J\sqrt{3}\angle 90° = I\angle 90°$$

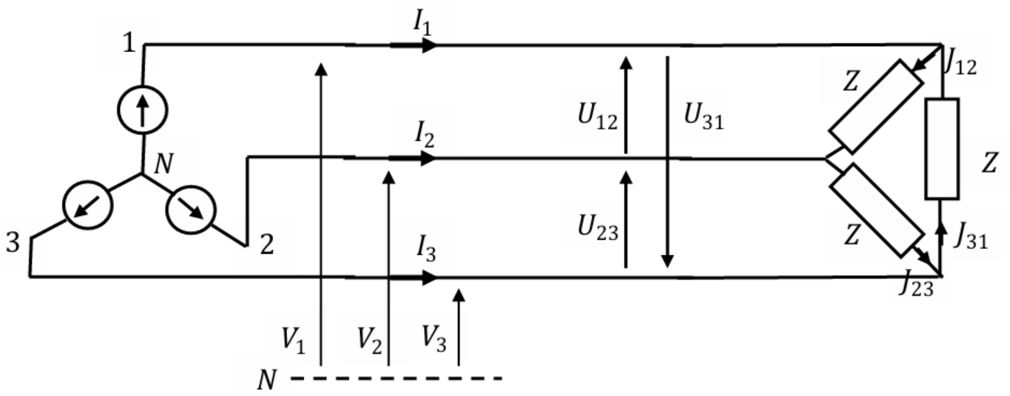


Fig. 1.21: Y-Δ connection

We easily state:

$$\overline{J_{ik}} = \frac{\overline{U_{ik}}}{\bar{Z}} \qquad i,k = 1,2,3 \qquad (1.31)$$

and we state in a similar way from nodes law (Fig. 1.22) :

$$\overline{I_1} = \sqrt{3}\,\overline{J_{\iota k}}\,\angle -30° \qquad (1.32)$$

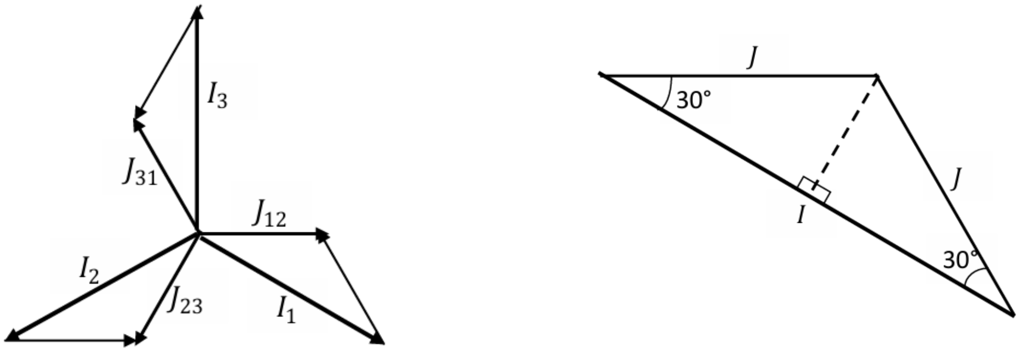


Fig. 1.22 : Representation of phase and line currents

But we have seen that:

$$\overline{U_{\iota k}} = \sqrt{3}\overline{V_\iota}\,\angle 30°$$

from where:

$$\overline{I_\iota} = \sqrt{3}\,\frac{\overline{U_{\iota k}}}{\bar{Z}}\,\angle -30° = \sqrt{3}\,\frac{\sqrt{3}\overline{V_\iota}\,\angle 30°}{\bar{Z}}\,\angle -30° = 3\,\frac{\overline{V_\iota}}{\bar{Z}} = \frac{\overline{V_\iota}}{\left(\frac{\bar{Z}}{3}\right)} \qquad (1.33)$$

which leads to the following equivalent single-phase diagram (Fig. 1.23):

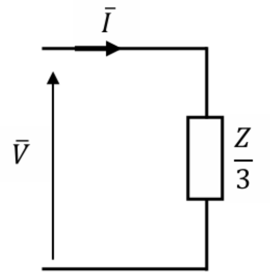


Fig. 1.23 : Equivalent single-phase diagram (Y-Δ case)

3) Δ-Y connection

The line voltages associated with a delta source can be expressed by phase voltages referenced to a fictitious neutral point N, such that $\overline{U_{\iota k}} = \sqrt{3}\overline{V_\iota}\,\angle 30°$. We obtain the simple equivalent single-phase diagram of Fig. 1.25.

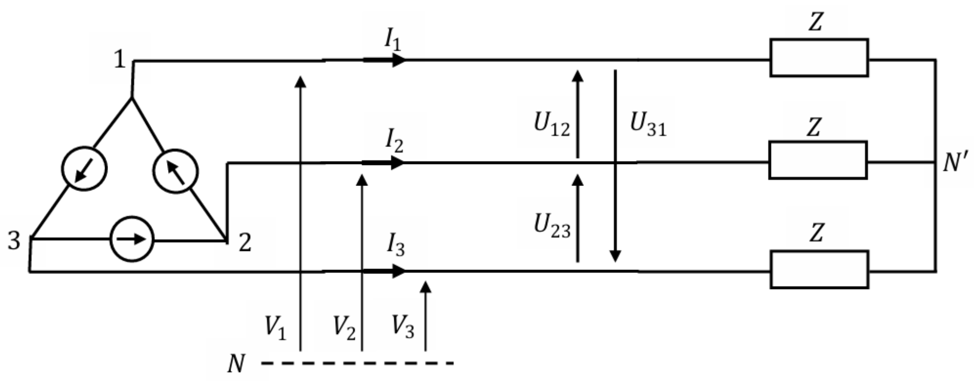


Fig. 1.24: Δ-Y connection

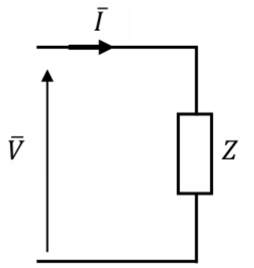


Fig. 1.25 : Equivalent single-phase diagram (Δ-Y case)

4) Δ-Δ connection

Δ-Δ connection mode can be deduced considering the Δ-Y coupling by replacing the triangular load with an equivalent star load of impedances Z/3 (Kennelly transformation). This leads to the equivalent single-phase diagram of figure 1.27.

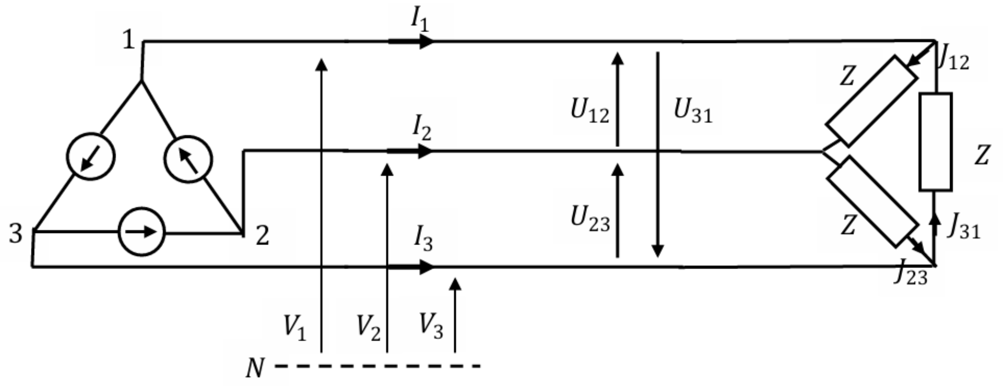


Fig. 1.26: Δ-Δ connection

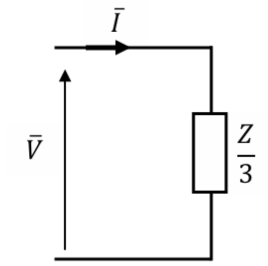


Fig. 1.27 : Equivalent single-phase diagram (Δ-Δcase)

Conclusion:

The study of a balanced and symmetrical three-phase network in sinusoidal mode is carried out using the equivalent single-phase diagram. It must only include phase voltages, line currents and impedances brought to star configuration.

6 Power in three phase systems

In three-phase systems, the powers of the three phases are considered simultaneously. Considering the phase voltages $v_1,\ v_2$ et v_3 and the line currents $i_1,\ i_2$ et i_3, we have as an expression for the instantaneous power:

$$p(t) = p_1 + p_2 + p_3 = v_1 i_1 + v_2 i_2 + v_3 i_3 \qquad (1.34)$$

In balanced sinusoidal three-phase operation and by choosing the reference appropriately, we can write:

$v_1(t) = V\sqrt{2}.cos(\omega t)$ $\quad i_1(t) = I\sqrt{2}.cos(\omega t + \varphi)$

$v_2(t) = V\sqrt{2}.cos(\omega t - 2\pi/3)$ $\quad i_2(t) = I\sqrt{2}.cos(\omega t - \frac{2\pi}{3} + \varphi)$

$v_3(t) = V\sqrt{2}.cos(\omega t - 4\pi/3)$ $\quad i_3(t) = I\sqrt{2}.cos(\omega t - \frac{4\pi}{3} + \varphi)$

Thus,

$$p(t) = 2VI\left[cos(\omega t)\,cos(\omega t + \varphi) + cos\left(\omega t - \frac{2\pi}{3}\right)cos\left(\omega t - \frac{2\pi}{3} + \varphi\right) + cos\left(\omega t - \frac{4\pi}{3}\right)cos(\omega t - \frac{4\pi}{3} + \varphi)\right] = 3VIcos\varphi \tag{1.35}$$

The power consumed by the load is constant. Unlike the average power in single-phase, no fluctuating power term appears:

$$p(t) = P = 3VIcos\varphi \tag{1.36}$$

This is one of the major advantages of three-phase systems, which allows the construction of rotating machines without power surges (constant motor torque).

Reactive power is the imaginary part of the complex power:

$$Q = 3VIsin\varphi \tag{1.37}$$

therefore

$$\bar{S} = P + jQ = 3VI\ \angle\varphi \tag{1.38}$$

The apparent power is equal to its magnitude

$$S = 3VI = \sqrt{P^2 + Q^2} \tag{1.39}$$

Or with line quantities:

$$P = \sqrt{3}UIcos\varphi \tag{1.40}$$

$$Q = \sqrt{3}UIsin\varphi \tag{1.41}$$

$$S = \sqrt{3}UI \tag{1.42}$$

This writing must however be handled with caution because the phase shift φ is measured between phase voltage and line current.