# BASIC MODELING AND PER-UNIT SYSTEM

## 1 Modeling of network components

The modeling and calculations that follow essentially concern the transmission network, but the methods described are independent of the voltage level. On such a network, the following types of fundamental elements are mainly distinguished: generation units, lines, power transformers, loads, and reactive power control devices.

Note that loads are only exceptionally customers directly connected in EHV. They more generally represent a connection point to the distribution network (typically 63 kV) via a transformer. Also note that the means of reactive power control (capacitors, reactors, static reactive power compensators) can be assimilated to loads or generators that consume or supply only reactive power.

In general, modeling a power network means above all making a certain number of simplifying assumptions that will determine both the complexity and the domain of validity of the model. The main assumptions retained, in the context of various basic studies, are as follows: only the behavior in steady state at 50 Hz is studied; the network is assumed to be linear. An important choice must then be made: is the calculation of power flows limited to totally balanced network operation; the study of the network can be carried out from an equivalent single-phase diagram. This approach is often sufficient in the context of operating a power network. It already makes it possible to predetermine, for a given generation plan and load level, what the loading of each line will be under normal operating conditions, and also what will be the network voltage profile.

### 1.1 Generators

The generator (turbo-alternator group) is considered the heart of the electrical network; it ensures the production of the electrical energy demanded by the consumer. In this study case, the generator is modeled by a constant voltage source which injects, at the node to which it is connected, an active power $P_G$ and a reactive power $Q_G$.

The alternator has two automatic control loops: one for automatic generation control (AGC: Automatic Generation Control) and the other for automatic voltage regulation (AVR: Automatic Voltage Regulator). The production of active energy in a generator is limited by

$$P_{Gmin} \leq P_G \leq P_{Gmax} \quad (1)$$

and the production of reactive energy is limited by

$$Q_{Gmin} \leq Q_G \leq Q_{Gmax} \quad (2)$$

This limitation is mainly due to the thermal limit of the stator and rotor windings as well as the limitation of the permissible rotor angle ($\delta \leq 30°$).

### 1.2 Loads

Loads are standard consumption devices. Load modeling plays a very important role in the study and analysis of network security. In the literature, there are two types of modeling of electrical loads, namely static modeling and dynamic modeling.

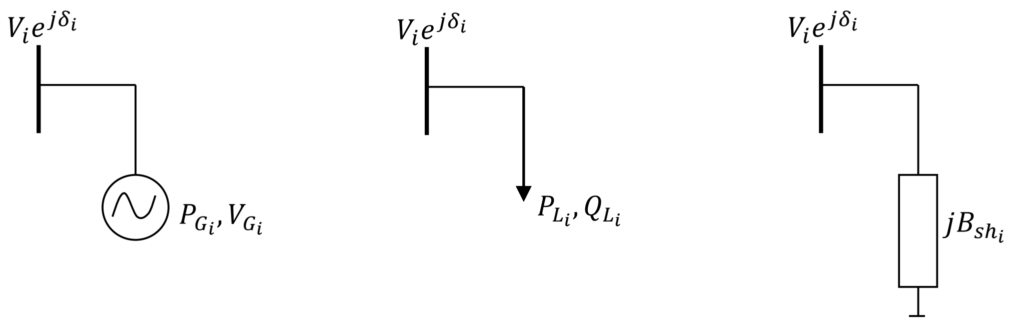
Dynamic modeling is relatively complicated; the power consumed by the load is a function of voltage and time; it is generally used for the study and analysis of transient stability. Static modeling is better suited to power flow programs. In this case, the load is represented by a constant active power and reactive power, $P_L$ and $Q_L$.

Reactive power may be supplied or absorbed, depending on whether the load is capacitive or inductive.

### 1.3 Shunt elements

Shunts are reactive elements which can absorb the reactive power of the system, but can also supply the system with reactive power. The reactive power $Q_{sh}$ injected by the shunt element is determined by the shunt admittance $B_{sh}$ and the voltage at that node.

The models of the previous elements (generator, load, shunt element) are represented in Fig. 1, connected to a node with voltage $V_i e^{j\delta_i}$.




(a) Generator (b) Load (c) Shunt element

Fig. 1: Elements that can be connected to a given bus

### 1.4 Lines

A transmission line connecting two nodes $i$ and $k$ is represented by its classical π-model. The series impedance of the line consists of the resistance $R$ and the inductive reactance $X$. Neglecting the shunt conductance, the shunt admittance corresponds to the shunt capacitance $B$.

It is important to keep in mind that a π-scheme can be a very accurate model of the behavior of an overhead line in steady-state balanced operation at 50 Hz, while being a very poor model for studying the behavior of this same line in other operating conditions, for example in high-frequency transient regime. Such a model cannot represent propagation phenomena along the line, such as those appearing during line energization or a lightning strike.

There is no absolutely good or bad model; there are only models that are, or are not, suited to the study of a given phenomenon. The $\pi$-model is well suited to the calculation of power flow in a transmission network.

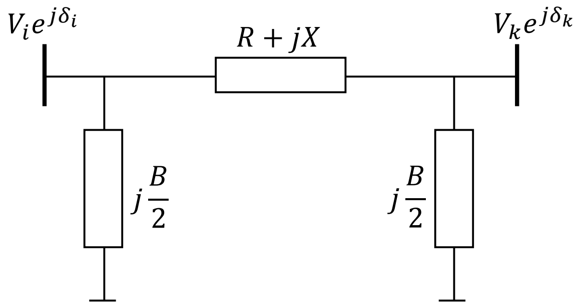



Fig. 2: π model of a line

**1.5 Transformers**

The power transformer is a fundamental element of electrical systems, making it possible to transmit energy economically with high efficiency and low voltage drop. Since power is proportional to the product of current and voltage, then for a given power it is possible to maintain a sufficiently low current level thanks to high voltages obtained using power transformers.

This makes it possible to reduce Joule losses and voltage drops. Currently, power transformers have an efficiency close to 100% and a power rating higher than 1300 MVA.

**1.5.1 Ideal transformer**

For an ideal transformer, it is assumed that:

- The windings have zero resistance (no Joule losses $RI^2 = 0$).
- The magnetic permeability $\mu_c$ of the ferromagnetic core is infinite, which corresponds to zero reluctance (lossless magnetic circuit).
- There is no leakage flux; the core flux $\Phi_c$ is entirely confined inside the magnetic circuit and links both windings.

A schematic representation of this transformer is given in Fig. 3, with primary $S_1, E_1, I_1$ and secondary $S_2, E_2, I_2$.

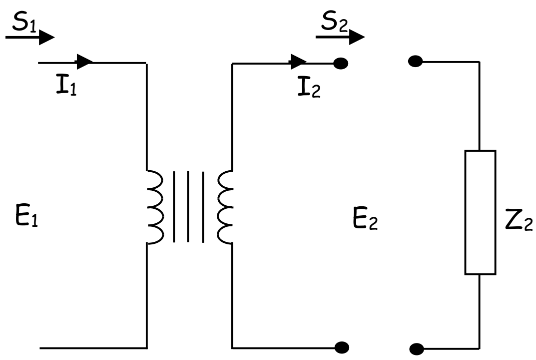



Fig. 3: Representation of an ideal transformer

The turns ratio is defined by

$$a_t = \frac{N_1}{N_2} \tag{3}$$

and the voltage and current relations are

$$\frac{E_1}{E_2} = \frac{N_1}{N_2} = a_t \quad \rightarrow \quad E_1 = a_t E_2 \tag{4}$$

$$\frac{I_1}{I_2} = \frac{N_2}{N_1} = \frac{1}{a_t} \quad \rightarrow \quad I_1 = \frac{I_2}{a_t} \tag{5}$$

The apparent power entering winding 1

$$S_1 = E_1 I_1^* = (a_t E_2)\left(\frac{I_2}{a_t}\right)^* = E_2 I_2^* = S_2 \tag{6}$$

is equal to the apparent power leaving winding 2, which means there are neither active nor reactive losses.

For an impedance $Z_2$ connected to the secondary,

$$Z_2 = \frac{E_2}{I_2} \tag{7}$$

the referred impedance to the primary

$$Z_2' = \frac{E_1}{I_1} = \frac{a_t E_2}{I_2/a_t} = a_t^2 Z_2 = \left(\frac{N_1}{N_2}\right)^2 Z_2 \tag{8}$$

### 1.5.2 Phase-shifting transformer

This is not a realization of a real transformer since it is materially impossible to obtain a complex transformation ratio. Nevertheless, it is used as a mathematical model to represent phase shift in three-phase transformers.

For this transformer, the complex ratio is defined as

$$a_t = 1e^{j\varphi} \tag{9}$$

where $\varphi$ is the phase shift angle

$$E_1 = a_t E_2 = e^{j\varphi} E_2 \tag{10}$$

$$I_1 = \frac{I_2}{a_t^*} = e^{j\varphi} I_2 \tag{11}$$

Thus $E_1$ leads $E_2$ by $\varphi$, and $I_1$ leads $I_2$ by $\varphi$, with unchanged magnitudes, and $S_1 = S_2$:

$$S_1 = E_1 I_1^* = (a_t E_2)\left(\frac{I_2}{a_t^*}\right)^* = E_2 I_2^* = S_2 \tag{12}$$

### 1.5.3 Real transformer

For the real transformer, active losses by Joule effect in winding 1 are represented by a resistance $R_1$. A leakage reactance $X_1$ in series accounts for leakage flux, which produces a voltage drop $jX_1 I_1$ proportional to $I_1$ and shifted by 90°, and reactive losses $X_1 I_1^2$. The same holds for winding 2.

For a finite permeability $\mu_c$ of the magnetic circuit, the total magnetomotive force is different from zero, defining a magnetizing current $I_m$ lagging $E_1$ by 90°, represented by a shunt susceptance $B_m$. In reality, there is an additional shunt branch represented by a conductance $G_c$, carrying the core loss current $I_c$ in phase with $E_1$. This gives the equivalent circuit in Fig. 4, with $R_1, X_1, R_2', X_2'$, $G_c$, and $-jB_m$.

According to this diagram, a real transformer operating in sinusoidal steady state is equivalent to an ideal transformer to which external branches of varying admittances and impedances have been connected. These external branches can be evaluated through short-circuit and no-load tests.

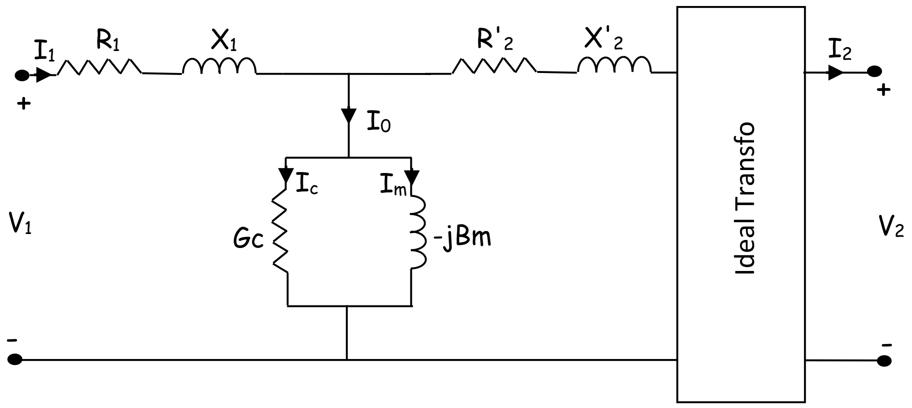



Fig. 4: Equivalent circuit of a real transformer

where:

$R_1, R_2$: winding resistances (active losses in the windings)

$X_1, X_2$: leakage reactances (leakage flux)

$B_m$: represents the magnetization current $I_m$

$G_c$: represents the core leakage current

If the shunt branch is omitted (neglecting excitation current), the equivalent circuit of Fig. 5 is obtained, generally acceptable for network studies except when efficiency or excitation phenomena are analyzed. For transformers of power higher than 500 kVA, winding resistances, being very small compared to reactances, are often neglected.

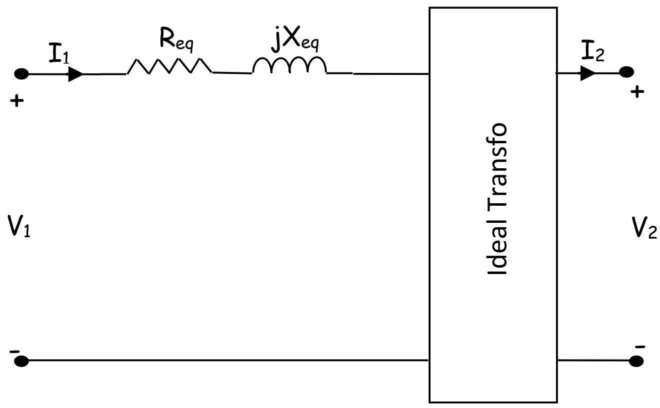



Fig. 5: Equivalent circuit if the excitation circuit is neglected

# 2 Per-unit system

## 2.1 Definitions

The elements of a power network operate at voltage levels where the kilovolt (kV) is the most suitable unit, while the megavolt-ampere (MVA) expresses apparent power. The per-unit system was introduced to express physical quantities such as current, voltage, power, and impedances as decimal fractions or multiples of base quantities.

In this system, the different voltage levels disappear and the electrical network, including synchronous generators, transformers, and lines, is reduced to a system of simple impedances. Thus, a quantity of the electrical system is expressed in relative units or in percentage with respect to a given base or reference quantity (per-unit, noted pu). The per-unit value is defined as:

$$per\ unit\ value = \frac{actual\ value}{base\ value}$$

All base values are only magnitude. They are not associated with any angle. The per unit values, however, are phasors. The phase angles of the currents and voltages and the power factor of the circuit are not affected by the conversion to per unit values. The per unit system values can also be expressed as per cent values.

For example, for a base voltage of 220 kV, a voltage of 210 kV equals $210/220 = 0.954$pu or 95.4%.

Among the advantages are:

- Reduced numerical values (especially impedances), then easier error detection.
- When values are expressed in pu, the comparison of electrical quantities with their "normal" values is straightforward. All quantities at network buses are close to 1 for appropriate bases.
- The factors $\sqrt{3}$ and 3 of three-phase systems disappear from formulas, and transformer diagrams are greatly simplified.

**Notes:**

- There is no angle conversion in pu,
- base value is always a real number.

To convert all quantities to pu, two independent base quantities are arbitrarily chosen at one point, generally $V_b$ and $S_b$. To keep electrical laws valid, we must have:

$$P_b = Q_b = S_b \tag{13}$$

$$I_b = \frac{S_b}{V_b} \tag{14}$$

$$Z_b = R_b = X_b = \frac{V_b}{I_b} = \frac{V_b^2}{S_b}\ ; \quad Y_b = \frac{1}{Z_b} \tag{15}$$

In addition, the following rules must be observed.

**Rule 1**: The base power is the same at every point of the electrical network.

**Rule 2**: For a transformer, the ratio of base voltages of primary and secondary equals the ratio of corresponding rated voltages:

$$\frac{V_{b1}}{V_{b2}} = \frac{V_{n1}}{V_{n2}} \tag{16}$$

### 2.2 Change of base

The conversion of per-unit impedances from old base values to new base values is given by:

$$Z_n(pu) = \frac{Z}{Z_{bn}} = \frac{Z_0(pu).Z_{b0}}{Z_{bn}} = Z_0(pu).\frac{V_{b0}^2/S_{b0}}{V_{bn}^2/S_{bn}} \tag{17}$$

where index $n$ refers to the new base and index 0 to the old base.

$$Z_n(pu) = Z_0(pu).\left(\frac{V_{b0}}{V_{bn}}\right)^2.\left(\frac{S_{bn}}{S_{b0}}\right) \tag{18}$$

### 2.3 Three-phase quantities

If $V_b$ is phase-to-neutral voltage and $U_b$ the line-to-line voltage, then:

$$S_{b1\emptyset} = \frac{S_{b3\emptyset}}{3} \quad ; \quad V_b = \frac{U_b}{\sqrt{3}} \tag{19}$$

Therefore:

$$I_b = \frac{S_{b3\emptyset}}{\sqrt{3}U_b} = \frac{S_{b1\emptyset}}{V_b} \tag{20}$$

And :

$$Z_b = \frac{V_b}{I_b} = \frac{V_b^2}{S_{b1\emptyset}} = \frac{U_b^2}{3S_{b1\emptyset}} = \frac{U_b^2}{S_{b3\emptyset}} = \frac{U_b}{\sqrt{3}I_b} \tag{21}$$

Also:

$$S_{b3\emptyset} = P_{b3\emptyset} = Q_{b3\emptyset} \tag{22}$$

$$U(pu) = \frac{U}{U_b} = \frac{V}{V_b} = V(pu) \tag{23}$$

The single-phase apparent power and the three-phase apparent power in pu

$$S_{1\emptyset}(pu) = \frac{S_{1\emptyset}}{S_{b1\emptyset}} = \frac{V.I^*}{V_b.I_b} = V(pu).I^*(pu) \tag{24}$$

$$S_{3\emptyset}(pu) = \frac{S_{3\emptyset}}{S_{b3\emptyset}} = \frac{3VI^*}{3V_bI_b} = V(pu).I^*(pu) = S_{1\emptyset}(pu) = S(pu) \tag{25}$$

For a given load $Z$, one obtains

$$Z = \frac{V}{I} = \frac{V}{\left(\frac{S_{charge3\emptyset}}{3V}\right)^*} = \frac{3V^2}{S_{charge3\emptyset}^*} = \frac{U^2}{S_{charge3\emptyset}^*} \tag{26}$$

$$Z(pu) = \frac{Z}{Z_b} = \frac{\frac{U^2}{S^*_{charge3\emptyset}}}{\frac{U_b^2}{S_{b3\emptyset}}} = \frac{\left(\frac{U}{U_b}\right)^2}{\frac{S^*_{charge3\emptyset}}{S_{b3\emptyset}}} = \frac{U^2(pu)}{S^*_{charge3\emptyset}} = \frac{V^2(pu)}{S^*_{charge1\emptyset}} \quad (27)$$

Thus, the quantities of single-phase and three-phase circuits are identical when converted to pu.

**2.4 Representation of transformers in pu**

For an ideal transformer, the per-unit voltages and currents satisfy:

$$E_1(pu) = \frac{E_1}{V_{b1}} = \frac{\frac{N_1}{N_2}E_2}{\frac{N_1}{N_2}V_{b2}} = \frac{E_2}{V_{b2}} = E_2(pu) \quad (28)$$

$$I_1(pu) = \frac{I_1}{I_{b1}} = \frac{\frac{N_2}{N_1}I_2}{\frac{N_2}{N_1}I_{b2}} = \frac{I_2}{I_{b2}} = I_2(pu) \quad (29)$$

This leads to the ideal transformer equivalent circuit in Fig. 6.

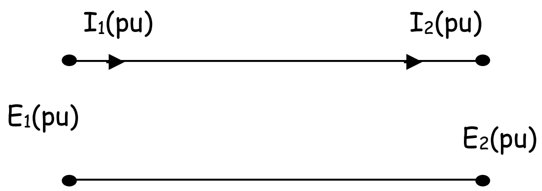



Fig. 6: Equivalent circuit of the ideal transformer in pu

For an impedance $Z_2$ connected to the secondary, the per-unit impedance is

$$Z_2(pu) = \frac{Z_2}{Z_{b2}} \quad with \ Z_{b2} = \frac{V_{b2}^2}{S_b} \quad (31)$$

The referred primary per-unit impedance is

$$Z'_2 = \left(\frac{N_1}{N_2}\right)^2 Z_2 \quad (30)$$

$$Z'_2(pu) = \frac{\left(\frac{N_1}{N_2}\right)^2 Z_2}{Z_{b1}} = \frac{\left(\frac{N_1}{N_2}\right)^2 Z_2}{\frac{V_{b1}^2}{S_b}} = \frac{\left(\frac{N_1}{N_2}\right)^2 Z_2}{\frac{\left(\frac{N_1}{N_2}V_{b2}\right)^2}{S_b}} = \frac{Z_2}{Z_{b2}} = Z_2(pu) \quad (32)$$

Thus, in pu, the secondary impedance referred to the primary equals the secondary winding impedance and vice versa.

The equivalent circuit of the real transformer in pu is shown in Fig. 7, with series elements $R_1(pu), jX_1(pu), R_2(pu), jX_2(pu)$ and shunt elements $-jB_m(pu)$ and $G_c(pu)$.

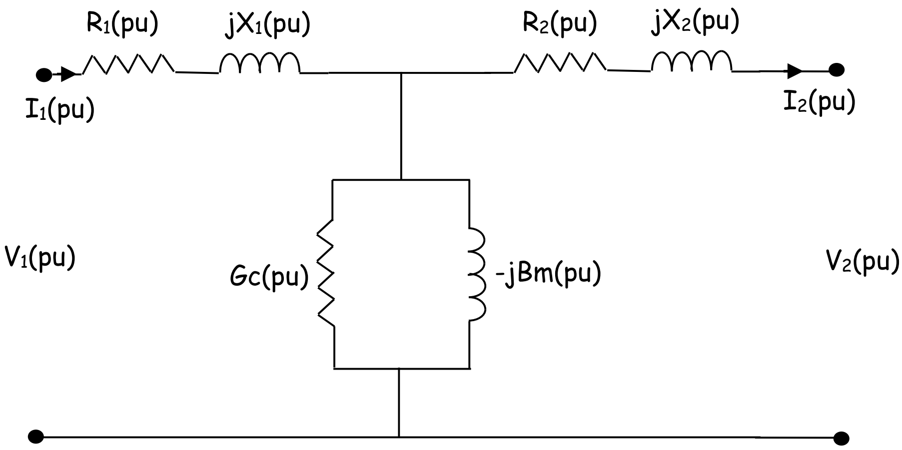



Fig. 7: Equivalent circuit of the actual transformer in pu

**2.5 Transformer with on-load tap changer**

The active power flow along a branch is controlled by the difference in voltage angles at its ends, while the reactive power flow is controlled by the difference in voltage magnitudes. Active and reactive powers can be adjusted by voltage regulating transformers and phase-shifting transformers.

The on-load tap-changing transformer (variable tap or OLTC) provides a degree of control of the electrical network by modifying the magnitudes and angles of voltages and currents by small amounts. For a given tap position, the transformer is modeled by two elements connected by a fictitious node (Fig. 8).

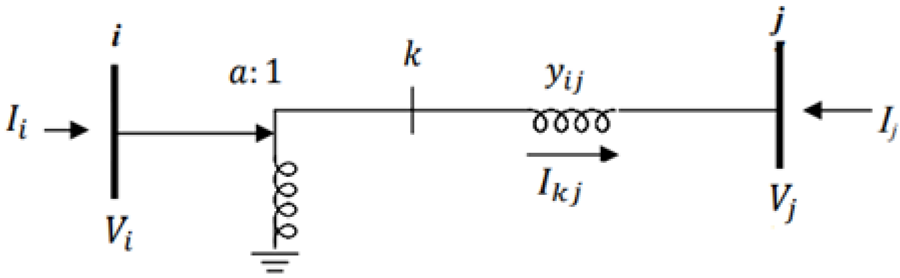



Fig. 8: Transformer with LTC

With series admittance $y_{ij}$ and complex ratio $a$, the currents and voltages satisfy:

$$\frac{V_i}{V_k} = \frac{I_{kj}}{I_i} \tag{33}$$

$$I_i = \frac{I_{kj}}{a} \tag{34}$$

$$I_{kj} = (V_k - V_j)y_{ij} \tag{35}$$

Therefore:

$$I_i = \frac{I_{kj}}{a} = (V_k - V_j)\frac{y_{ij}}{a} \tag{36}$$

And, since $V_k = V_i/a$,

$$I_i = (V_i - aV_j)\frac{y_{ij}}{a^2} \tag{37}$$

$$I_j = (V_j - V_k)y_{ij} = (aV_j - V_i)\frac{y_{ij}}{a} \tag{38}$$

In matrix form:

$$\begin{bmatrix} I_i \\ I_j \end{bmatrix} = \begin{bmatrix} \frac{y_{ij}}{a^2} & -\frac{y_{ij}}{a} \\ -\frac{y_{ij}}{a} & y_{ij} \end{bmatrix} \begin{bmatrix} V_i \\ V_j \end{bmatrix} \tag{39}$$

This can be represented by an equivalent $\pi$-circuit with series branch $A$ and shunt branches $B$ and $C$ (Fig. 9).

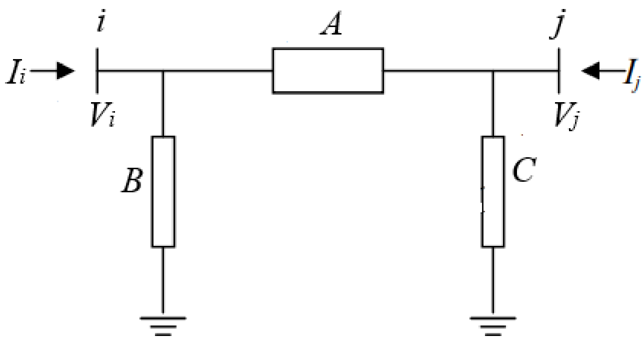



Fig. 9: Transformer represented as a dipole

where:

$$\begin{aligned} I_i &= (V_i - V_j)A + V_iB \\ I_j &= (V_j - V_i)A + V_jC \end{aligned} \tag{40}$$

By identification :

$$\begin{aligned} A &= \frac{y_{ij}}{a} \\ B &= \frac{1}{a}\left(\frac{1}{a} - 1\right)y_{ij} \\ C &= \left(1 - \frac{1}{a}\right)y_{ij} \end{aligned} \tag{41}$$

Fig. 10 summarizes the model of a variable tap transformer.

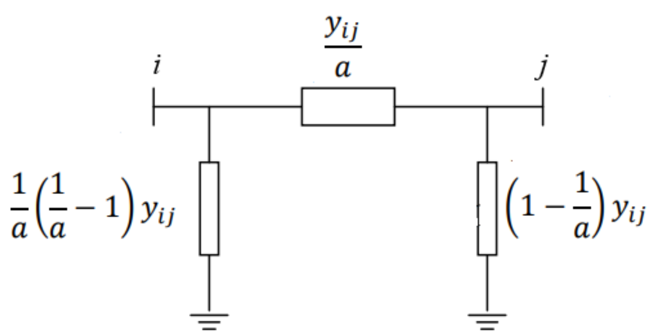



Fig. 10: LTC transformer model

### 2.6 Per-unit analysis method

Circuit analysis using pu values can be summarized as:

- Choose a common three-phase MVA base $S_b$.
- In an arbitrary part of the system, choose a line-to-line voltage base $U_b$.
- Convert line voltages through transformers according to their nameplate ratios.
- Find the base impedances in the various sections using $Z_b = U_b^2/S_b$.
- Adjust nameplate per-unit impedances according to the new bases using the change-of-base formula.
- Draw the impedance diagram for the whole system and solve for the desired quantities.
- Convert back to real values if necessary.

### 2.7 Example

Choosing a base apparent power of 10MVA and a base line voltage of 69 kV (load side), give the pu equivalent circuit of the electrical system in Fig. 11 and then calculate the load current in pu and in amperes.

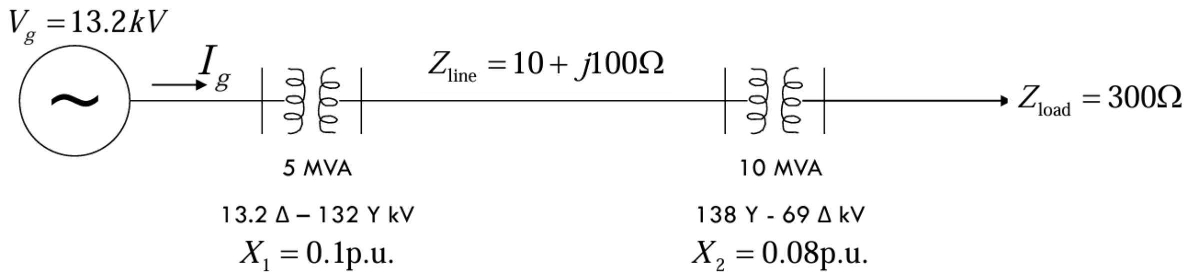



Fig. 11: Typical electric power system diagram

Base values of power and voltages:

$$S_b = 10MVA \qquad V_{b3} = 69kV$$

$$V_{b2} = \frac{138}{69} \times 69 = 138kV$$

$$V_{b3} = \frac{13.2}{132} \times 138 = 13.8kV$$

For zone 1 :

$$Z_{b1} = \frac{(U_{b1})^2}{S_b} = \frac{(13.8)^2}{10} = 19.04\ \Omega$$

$$I_{b1} = \frac{S_b}{\sqrt{3}U_{b1}} = \frac{10.\,10^3}{\sqrt{3} \cdot 13.8} = 418.4\ A$$

$$V_g(pu) = \frac{V_g}{V_{b1}} = \frac{13.2}{13.8} = 0.96\ pu$$

$$X_1 = 0.1 \times \frac{10}{5} \times \left(\frac{13.2}{13.8}\right)^2 = 0{,}183\ pu$$

For zone 2:

$$Z_{b2} = \frac{(U_{b2})^2}{S_b} = \frac{(138)^2}{10} = 1904\ \Omega$$

$$I_{b2} = \frac{S_b}{\sqrt{3}U_{b2}} = \frac{10.10^3}{\sqrt{3}\cdot 138} = 41.84\ A$$

$$Z_{line}(pu) = \frac{Z_{line}}{Z_{b2}} = \frac{10 + j100}{1904} = 5.25\cdot 10^{-3}(1 + j10)\ pu$$

For zone 3 :

$$Z_{b3} = \frac{(U_{b3})^2}{S_b} = \frac{(69)^2}{10} = 476\ \Omega$$

$$I_{b3} = \frac{S_b}{\sqrt{3}U_{b3}} = \frac{10.10^3}{\sqrt{3}\cdot 69} = 83.67\ A$$

$$X_2 = 0.08 \times \frac{10}{10} \times \left(\frac{69}{69}\right)^2 = 0.08\ pu$$

$$Z_{load}(pu) = \frac{Z_{load}}{Z_{b3}} = \frac{300}{476} = 0.63\ pu$$

The pu equivalent diagram is shown in Fig. 12.

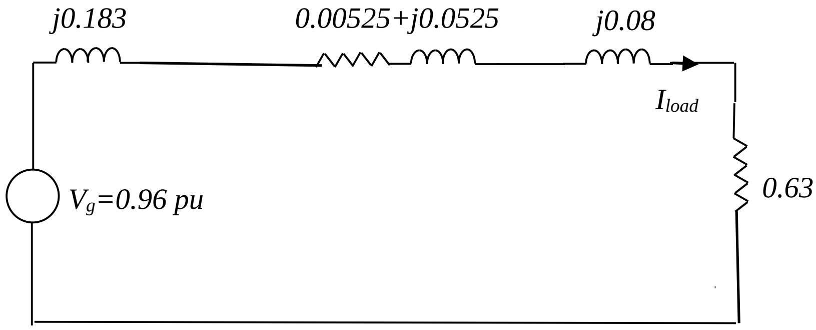



Fig. 12: Equivalent circuit in pu

The total impedance

$$Z_T = j0.183 + 0.00525 + j0.0525 + j0.08 + 0.63 = 0.63525 + j0.3155 = 0.7093\angle 26.41°\ pu$$

The load current is

$$I_{load} = \frac{V_g}{Z_T} = \frac{0.96\angle 0°}{0.7093\angle 26.41°} = 1.353\angle -26.41°\ pu$$

or in amperes

$$I_{load} = (1.353\angle -26.41°) \times 83.67 = 113.2\angle -26.41°\ A$$