CALCULATION OF ELECTRICAL NETWORK MATRICES

1. Introduction

Mathematical formulation or modeling is the first step in the analysis of an electrical network. The model must describe the characteristics of the individual components of the whole power system as well as the relationships governing the interconnection of these elements.

A matrix equation of the electrical network provides a suitable model for solutions using a computer. The elements of a network matrix depend on the choice of independent variables, which may be currents or voltages. The elements of the matrix are therefore either impedances or admittances.

The form of the matrix depends on the reference structure, that is, the nodes or the meshes.

2. Graph Theory

The **graph** is a geometric drawing that illustrates connections graphically. Its representation is associated with a table that presents the properties of binary matrices. The elements of these matrices are equal to 0 or 1, with possibly an indication (sign +/−) when the graph is known to have a direction, as in the general case of electrical networks.

To describe the geometric structure of a network, it is sufficient to replace the network components with simple segments without considering their characteristics. These line **segments** are called **elements**, and their endpoints are called **vertices** or **nodes**.

A node and a segment are **incident** if the node is an endpoint of the segment. Nodes can be incident with more than one element.

The graph of an electrical network shows the geometric interconnection of the network elements. A **subgraph** is a subset of the elements of the graph.

A **path** is a subgraph of connected elements with no more than 2 elements connected to a single node.

A graph is **connected** if and only if there is a path between every pair of nodes. If a direction is assigned to each element of a connected graph, the graph is said to be **oriented**. In electrical systems, this direction corresponds to the direction of current flow in the element represented by the segment. An exception is made for independent voltage sources, to which the opposite direction of the current must be assigned.

A **tree** is a subgraph that contains all the nodes of a graph but no closed path (mesh). The segments of this tree are called **branches**. The remaining segments are called the **links**. The set of links forms the **co-tree**.

If a link is added to a tree, the resulting graph contains a closed path called a **circuit**. The addition of each subsequent link forms one or more additional circuits. Circuits containing only one link are independent and are called **fundamental circuits** or **basic loops**. The orientation of a fundamental circuit is chosen similar to that of the link.

Finally, a reference node must be chosen, and it is common practice to assign this node the number zero (0).

Example 1:

Consider an electrical network with 4 nodes (bus bars) with its equivalent single-phase circuit (Fig. 1). The graph representation (Fig. 2) shows the nodes and elements with their orientations. The branches are numbered 1, 2, 3, 4 and the links are numbered 5, 6, 7.

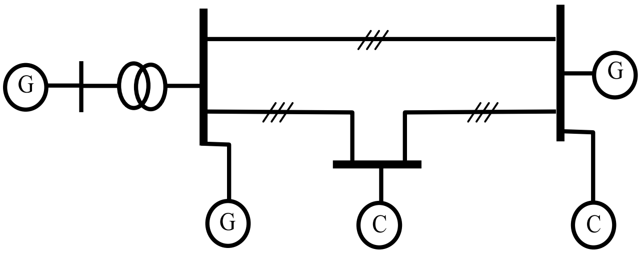


Fig. 1a: Three-phase electrical network

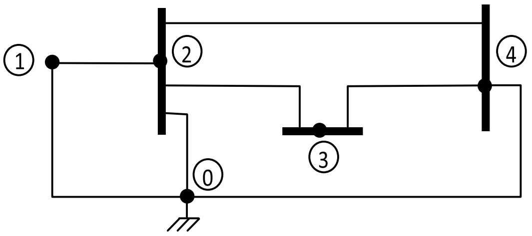


Fig. 1b: Single-phase equivalent circuit

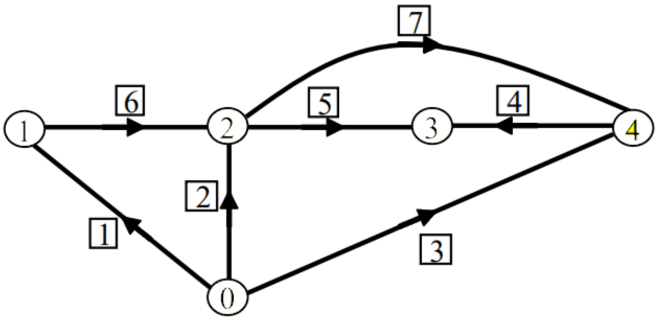


Fig. 2: Representation of the network by a graph

Branches: 1, 2, 3, 4 Links: 5, 6, 7

3. Matrix Representation of Networks

The formulation of network equations is based on the definition of an exact and coherent mathematical model that describes the characteristics of individual components (machines, lines, transformers, and loads) and the interconnection between these components. The matrix equation is a model suitable for mathematical processing and transformation from a systems perspective. The elements of the matrix can be impedances (bus voltages written as a function of injected currents) or admittances (injected currents written as a function of bus voltages).

The network can be described by three types of matrices:

- **Elementary matrices** (or **primitive matrices**): these describe individual components, taking account, where applicable, of their electromagnetic coupling (capacitive and inductive). They have a diagonal structure, with component coupling represented by off-diagonal elements.
- **Incidence matrices**: these describe the interconnections between the different components of the network. The values of these matrices are binary: 1, 0, −1, representing the link between the branches and the nodes of the network with their orientation.

- **Transfer matrices**: these mathematically describe the electrical behavior of the meshed network. They are essentially the impedance or admittance matrices corresponding to the nodes of the network (nodal matrices).

4. Incidence Matrix

Incidence matrices characterize the relationship between the elements of the network (commonly called branches) and the connecting nodes between these elements.

4.1 Nodal Incidence Matrix

The nodal incidence matrix A describes the topology of the system without regard to the nature of its component elements. For a graph with n nodes and e segments, the matrix A is formed with n columns and e rows.

Its elements a_{ij}can take the values 0, −1, or 1 as follows:

$a_{ij} = 1$ if element i is incident with node j and oriented away from this node;

$a_{ij} = -1$ if element i is incident with node j and oriented toward this node;

$a_{ij} = 0$ if element i is not incident with node j.

For the previous example, matrix A is as follows:

$e\,/\,n$	0	1	2	3	4
1	1	-1	0	0	0
2	1	0	-1	0	0
3	1	0	0	0	-1
4	0	0	0	-1	1
5	0	0	1	-1	0
6	0	1	-1	0	0
7	0	0	1	0	-1

For each row i:

$$\sum_{j=0}^{n-1} a_{ij} = 0 \tag{1}$$

Indeed, in the same row corresponding to the branch in question, there are only two nonzero elements: the first corresponds to the starting node with the value +1, and the second corresponds to the ending node with the value −1.

4.2 Reduced Incidence Matrix

Instead of the nodal incidence matrix, one generally uses the **reduced nodal incidence matrix**, which corresponds to the nodal incidence matrix in which the choice of a reference node for voltages leads to the removal of one column from matrix A (generally the first).

5. Primitive Network Matrices

The primitive network represents all the components of the network, including their electrical and magnetic couplings. Each component is defined by its impedance z_{pq} or admittance $y_{pq} = 1/z_{pq}$ with indices p (starting node) and q (ending node).

Furthermore, generators are modeled by an electromotive force (emf) e_{pq} in series with an internal impedance (Thevenin model), or by a current source J_{pq} with an internal admittance in parallel (Norton model).

5.1 Equation in Terms of Impedance

The voltage across the terminals V_{pq} is related to the current I_{pq} by the emf e_{pq} and the impedance z_{pq} as follows (Fig. 3):

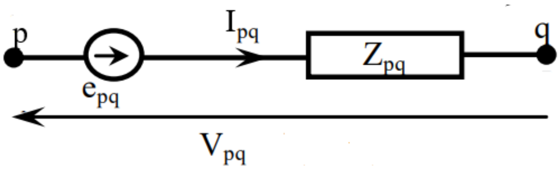


Fig. 3: Representation of the voltage source with its internal impedance

$$V_{pq} + e_{pq} = z_{pq}.I_{pq} \tag{2}$$

5.2 Equation in Terms of Admittance

The Fig. 4 shows equivalent Norton circuit of the generator (current source in parallel with the internal admittance).

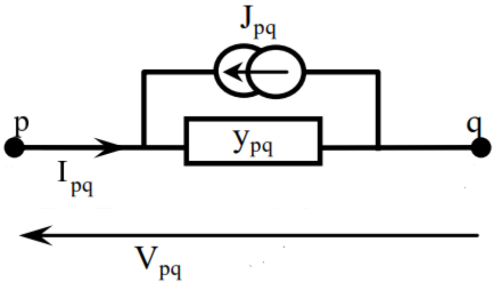


Fig. 4: Representation of the current source with its internal admittance

The currents I_{pq}, J_{pq}, and the voltage across the generator terminals V_{pq} are related by the equations:

$$I_{pq} + J_{pq} = y_{pq} V_{pq} \tag{3}$$

$$J_{pq} = -y_{pq} e_{pq} \tag{4}$$

The primitive network matrix is a matrix with diagonal dominance where the elements correspond to the impedances of each link in the network. These impedances are the self-impedances $z_{pq,pq}$ and the coupling impedances between links pq and rs, which represent the off-diagonal elements $z_{pq,rs}$ (Fig. 5).

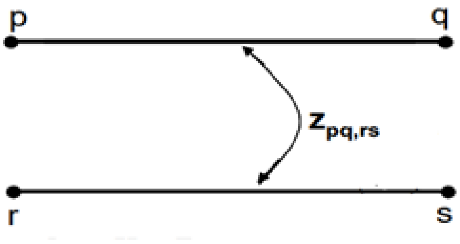


Fig. 5: Coupled elements

The current vector I, voltage vector V, and impedance matrix z of the primitive network are expressed by:

$$I = \begin{bmatrix} I_{12} \\ I_{13} \\ \vdots \\ I_{23} \\ \vdots \\ I_{pq} \\ \vdots \end{bmatrix} \qquad V = \begin{bmatrix} V_{12} \\ V_{13} \\ \vdots \\ V_{23} \\ \vdots \\ V_{pq} \\ \vdots \end{bmatrix} \qquad z = \begin{bmatrix} \ddots & & \vdots & & \vdots & \\ & \ddots & \vdots & & \vdots & \\ & & \ddots & & z_{rs,pq} & \\ & & \vdots & \ddots & \vdots & \\ \cdots & \cdots & z_{pq,rs} & \cdots & z_{pq,pq} & \cdots \\ & & \vdots & & & \end{bmatrix} \tag{5}$$

The vectorial representation of the current sources J and the emfs e, as well as the equations of the elementary network are:

$$e = \begin{bmatrix} e_{12} \\ e_{13} \\ \vdots \\ e_{23} \\ \vdots \\ e_{pq} \\ \vdots \end{bmatrix} \qquad J = \begin{bmatrix} J_{12} \\ J_{13} \\ \vdots \\ J_{23} \\ \vdots \\ J_{pq} \\ \vdots \end{bmatrix} \qquad V + e = zI \quad I + J = yV \quad y = z^{-1} \tag{6}$$

Example 2 :

Consider the network of Fig. 6 with 4 nodes and 5 segments. Lines 1 and 4 (both connected in parallel between nodes 1 and 2) are mutually coupled. They are designated by indices (1) and (2) respectively.

The network is presented in a connection table (Table 1) showing the values of the line impedances and the mutual coupling impedances. The impedances are given in relative units. Using the link classification defined in Table 1, the z -impedance matrix of the primary network can be constructed.

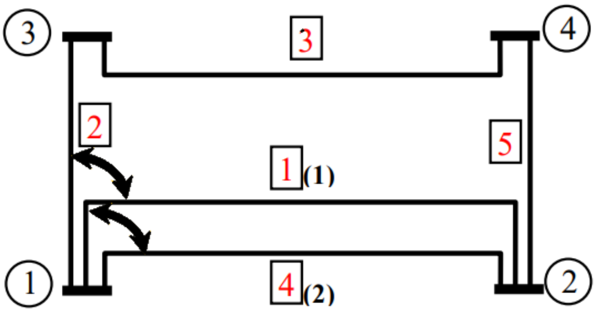


Fig. 6: Example of a primary network

Table 1: Self and mutual impedances

line	nodes	self	nodes	mutual
1	1-2 (1)	0.6		
2	1-3	0.5	1-2 (1)	0.1
3	3-4	0.5		
4	1-2 (2)	0.4	1-2 (1)	0.2
5	2-4	0.2		

An appropriate choice of line numbering allows better use of the impedance matrix z to transform it into diagonal sub-matrices, particularly for determining its inverse.

By permuting links 3 and 4, the matrix can be rearranged to have a block-diagonal structure. The inverse of the matrix is then obtained by inverting a 3×3 matrix and a diagonal 2×2 matrix separately.

$$z = \begin{array}{c|ccccc} \text{lines} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0{,}6 & 0{,}1 & & 0{,}2 & \\ 2 & 0{,}1 & 0{,}5 & & & \\ 3 & & & 0{,}5 & & \\ 4 & 0{,}2 & & & 0{,}4 & \\ 5 & & & & & 0{,}2 \end{array}$$

Fig. 7a : Primitive impedance matrix

$$z = \begin{array}{c|ccccc} \text{lines} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0{,}6 & 0{,}1 & 0{,}2 & & \\ 2 & 0{,}1 & 0{,}5 & & & \\ 3 & 0{,}2 & & 0{,}4 & & \\ 4 & & & & 0{,}5 & \\ 5 & & & & & 0{,}2 \end{array}$$

Fig. 7b : Primitive impedance matrix with permutation of links 3 and 4

$$y = z^{-1} = \begin{array}{c|ccccc} \text{lines} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 2{,}08 & -0{,}42 & -1{,}04 & & \\ 2 & 0{,}1 & 0{,}5 & 0{,}21 & & \\ 3 & -1{,}04 & 0{,}21 & 0{,}4 & & \\ 4 & & & & 0{,}5 & \\ 5 & & & & & 0{,}2 \end{array}$$

Fig. 7c: Inverse z-matrix of Fig. 7b

5.3. Transfer Matrices

Consider a network with $(n+1)$ nodes, numbered $0,1,2,\dots,n$. Let node 0 be the reference node to which all bus voltages $V_1,\dots,V_n$ are indexed, and $I_1,\dots,I_n$ the currents injected at these nodes.

This allows replacing the internal structure by transfer matrices relating bus currents and voltages (Fig. 8).

Define vectors

$$V_{bus}=\begin{bmatrix}V_1\\V_2\\\vdots\\V_i\\\vdots\\V_n\end{bmatrix} \text{ and } I_{bus}=\begin{bmatrix}I_1\\I_2\\\vdots\\I_i\\\vdots\\I_n\end{bmatrix}$$

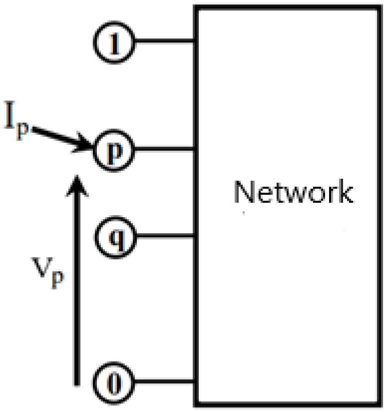


Fig. 8 : Network with bus voltages and injected currents

The network operation is then modeled through the bus impedance matrix Z_{bus} or the bus admittance matrix Y_{bus} by the equations:

$$I_{bus}=Y_{bus}.V_{bus} \quad (7)$$

$$V_{bus}=Z_{bus}.I_{bus} \quad (8)$$

Example 3 :

The network in Fig. 9a below can be modified and represented by a diagram as shown in Fig. 9b.

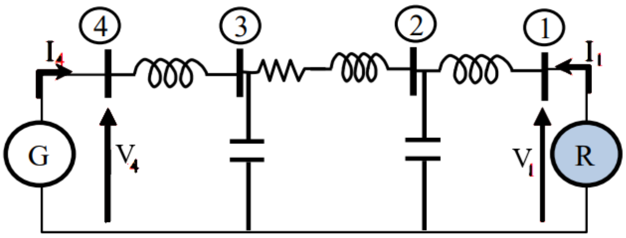


Fig. 9a: Example of a 4-bus network

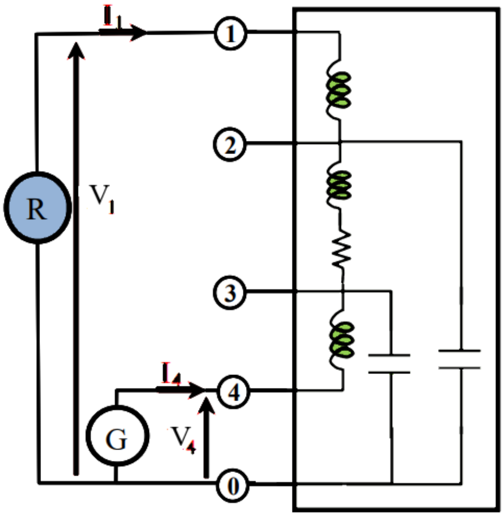


Fig. 9b: Schematic modification of the system in Fig. 9a

6. Formation of the Bus Admittance Matrix

The matrices of particular interest in the analysis of an electrical network are the nodal impedance and admittance matrices. The admittance matrix is the easiest to compute, and its modification in case of topology changes is also simple.

There are two known methods for determining the admittance matrix Y_{bus} of an electrical network. One uses the nodal incidence matrix A; the other is based on a simple formulation related to appropriate modeling of the different network components.

6.1 From Network Element Admittances and Incidence Matrix

Consider the matrix equation of the primitive network:

$$I + J = yV \tag{9}$$

Multiplying by the transpose of the incidence matrix A^t:

$$A^t.(I + J) = A^t.y.V \ \Rightarrow \ A^t.I + A^t.J = A^t.y.V \tag{10}$$

The first term $A^t.I$ is the sum of currents arriving at each node of the network, which is equal to zero (Kirchhoff's law). The term $A^t.J$ is the sum of currents injected into each node, equal to I_{bus}. Thus:

$$I_{bus} = A^t.y.V \tag{11}$$

The total complex power injected into the network is the same whether expressed in primitive form or in nodal form:

$$S = (I_{bus})^{*t}.V_{bus} = J^{*t}.V \tag{12}$$

But since

$$I_{bus} = A^t.J \tag{13}$$

then,

$$(I_{bus})^{*t} = (A^t.J)^{*t} \tag{14}$$

Since matrix A is composed of real numbers, then $A = A^*$ and consequently:

$$(I_{bus})^{*t} = J^{*t}.A \tag{15}$$

Therefore:

$$J^{*t}.A.V_{bus} = J^{*t}V \tag{16}$$

Thus:

$$V = A.V_{bus} \tag{17}$$

Considering the relation (11) :

$$I_{bus} = A^t yAV_{bus} = Y_{bus}.V_{bus} \tag{18}$$

In conclusion :

$$Y_{bus} = A^t yA \tag{19}$$

6.2 From Network Element Admittances

This method is the simplest and most convenient, as it requires fewer operations, especially when the network has no mutual coupling between its elements. For the sake of simplicity, we will denote the bus admittance matrix simply by Y.

For a network with n independent nodes and one reference node (0), for a node i among the n nodes different from the reference node:

$$I_i = \sum_{j=1}^{n} I_{ij} \tag{20}$$

where:

I_i: is the current injected at node i by an external source (generator or load)

I_{ij}: is the current from node i to j through branch $(i - j)$ (The branches represent transmission lines and transformers). We also have:

$$I_{ij} = (V_i - V_j)y_{ij} \tag{21}$$

where:

V_i and V_j: voltages at nodes i and j respectively

y_{ij}: self-admittance of branch $(i - j)$

Therefore:

$$I_i = \sum_{i=1}^{n} y_{ij}(V_i - V_j) = y_{i0}(V_i - V_0) + y_{i1}(V_i - V_1) + y_{i2}(V_i - V_2) + \cdots + y_{in}(V_i - V_n) \tag{22}$$

Where $V_0 = 0$ and $i = 1 \ldots n$

and y_{i0} the sum of shunt admittances to ground at node i:

$$I_i = V_i\left[y_{i0} + \sum_{j=1}^{n} y_{ij}\right] - y_{i1}V_1 - y_{i2}V_2 - \cdots - y_{in}V_n \quad i = 1,2,\ldots,n \tag{23}$$

the nodal equations can be written in matrix form: $I = YV$

$$\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_i \\ \vdots \\ I_n \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1i} & \cdots & Y_{1n} \\ Y_{21} & Y_{22} & \cdots & Y_{2i} & \cdots & Y_{2n} \\ \cdots & \cdots & \ddots & \cdots & \ddots & \cdots \\ Y_{i1} & Y_{i2} & \cdots & Y_{ii} & \cdots & Y_{in} \\ \cdots & \cdots & \ddots & \cdots & \ddots & \cdots \\ Y_{n1} & Y_{n2} & \cdots & Y_{ni} & \cdots & Y_{nn} \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_i \\ \vdots \\ V_n \end{bmatrix} \tag{24}$$

The bus admittance matrix elements are:

$$\begin{cases} Y_{ii} = y_{i0} + \sum_{j=1}^{n} y_{ij} \\ Y_{ij} = -y_{ij} \end{cases} \tag{25}$$

The matrix Y is square of order n, symmetric, and sparse for large networks.

Algorithm for computing Y:

Read branch and node data

Convert all impedances to admittances : $y_{ij} = -\frac{1}{z_{ij}}$

For each node i, compute diagonal element: $Y_{ii} = \sum_{j=1}^{n} y_{ij} \quad j \neq i$

For each pair of nodes $i \neq j$, compute off-diagonal element: $Y_{ij} = -y_{ij}$

The admittance matrix is symmetric: $Y_{ij} = Y_{ji}$

Example 4:

For a 4-bus network, write Kirchhoff's current equations, rearrange them, and assemble the bus admittance matrix Y. After substituting numerical admittances, the full complex matrix Y is obtained.

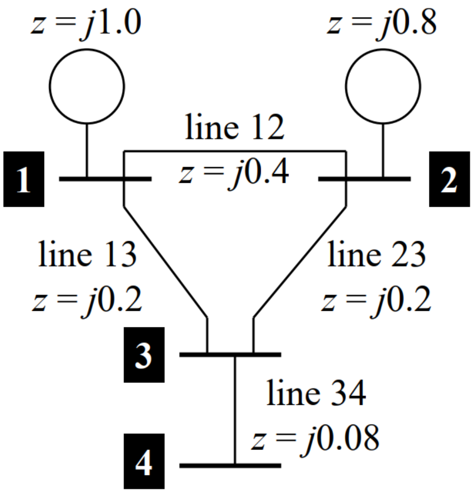
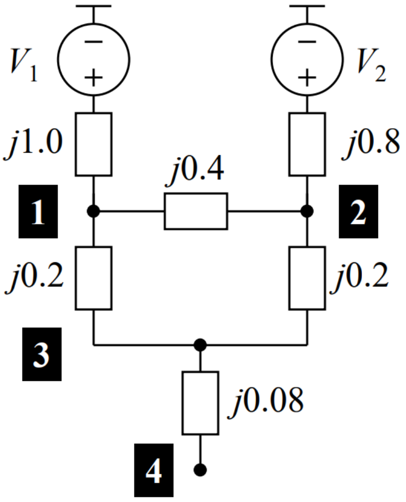


Fig. 10a: 4-bus network



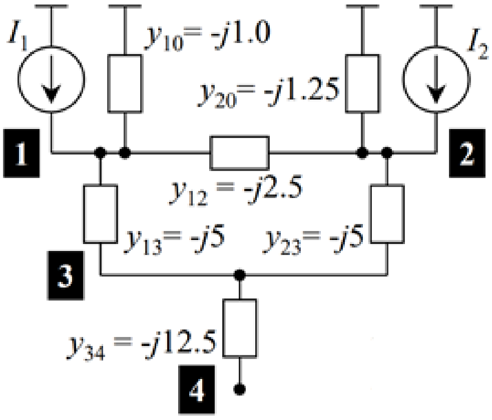


Fig. 10b: Representation by impedances

Fig. 10c: Representation by admittances

Kirchhoff's equations are written as:

$$I_1 = y_{10}V_1 + y_{12}(V_1 - V_2) + y_{13}(V_1 - V_3)$$

$$I_2 = y_{20}V_2 + y_{21}(V_2 - V_1) + y_{23}(V_2 - V_3)$$

$$0 = y_{31}(V_3 - V_1) + y_{32}(V_3 - V_2) + y_{34}(V_3 - V_4)$$

$$0 = y_{43}(V_4 - V_3)$$

By rearranging these equations:

$$I_1 = (y_{10} + y_{12} + y_{13})V_1 - y_{12}V_2 - y_{13}V_3$$

$$I_2 = -y_{21}V_1 + (y_{20} + y_{21} + y_{23})V_2 - y_{23}V_3$$

$$0 = -y_{31}V_1 - y_{32}V_2 + (y_{31} + y_{32} + y_{34})V_3 - y_{34}V_4$$

$$0 = -y_{43}V_3 + y_{43}V_4$$

The matrix formulation of these equations:

$$\begin{bmatrix} I_1 \\ I_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (y_{10} + y_{12} + y_{13}) & -y_{12} & -y_{13} & 0 \\ -y_{21} & (y_{20} + y_{21} + y_{23}) & -y_{23} & 0 \\ -y_{31} & -y_{32} & (y_{31} + y_{32} + y_{34}) & -y_{34} \\ 0 & 0 & -y_{43} & y_{43} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}$$

With the evaluation of the matrix entries:

$$Y_{11} = (y_{10} + y_{12} + y_{13}) = -j8{,}50 \qquad Y_{12} = Y_{21} = -y_{12} = j2{,}50$$

$$Y_{13} = Y_{31} = -y_{13} = j5{,}00 \qquad Y_{22} = (y_{20} + y_{21} + y_{23}) = -j8{,}75$$

$$Y_{23} = Y_{32} = -y_{23} = j5{,}00 \qquad Y_{33} = (y_{31} + y_{32} + y_{34}) = -j22{,}50$$

$$Y_{34} = Y_{43} = -y_{34} = j12{,}50 \qquad Y_{44} = y_{34} = -j12{,}50$$

We obtain:

$$\begin{bmatrix} I_1 \\ I_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -j8{,}50 & j2{,}50 & j5{,}00 & 0 \\ j2{,}50 & -j8{,}75 & j5{,}00 & 0 \\ j5{,}00 & j5{,}00 & -j22{,}50 & j12{,}50 \\ 0 & 0 & j12{,}50 & -j12{,}50 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}$$

Therefore:

$$Y = \begin{bmatrix} -j8{,}50 & j2{,}50 & j5{,}00 & 0 \\ j2{,}50 & -j8{,}75 & j5{,}00 & 0 \\ j5{,}00 & j5{,}00 & -j22{,}50 & j12{,}50 \\ 0 & 0 & j12{,}50 & -j12{,}50 \end{bmatrix}$$

7. Formation of the Bus Impedance Matrix

The matrix Z_{bus} contains driving-point impedances (diagonal) and transfer impedances (off-diagonal) between nodes and reference. It can be obtained as the inverse of the nodal admittance matrix:

$$Z_{bus} = (Y_{bus})^{-1} \tag{26}$$

but a direct constructive method is often preferred.
The constructive method starts from an initial partial network with a known Z_{bus} (possibly a single element to the reference), then adds elements one by one. Let $p - q$ be an element that is added to the partial network of m nodes; this element can be either a branch (Fig. 11), or a loop (Fig. 12).

For each added element $p - q$:

- If it is a **branch**: a new node q is added, increasing matrix size from $m \times m$ to $(m + 1) \times (m + 1)$. Only the new row and column need to be computed.
- If it is a **loop**: no new node; matrix order remains m, but all entries are updated to account for the loop.

7.1. Adding a branch

When adding a branch $p - q$, the new node q is created.

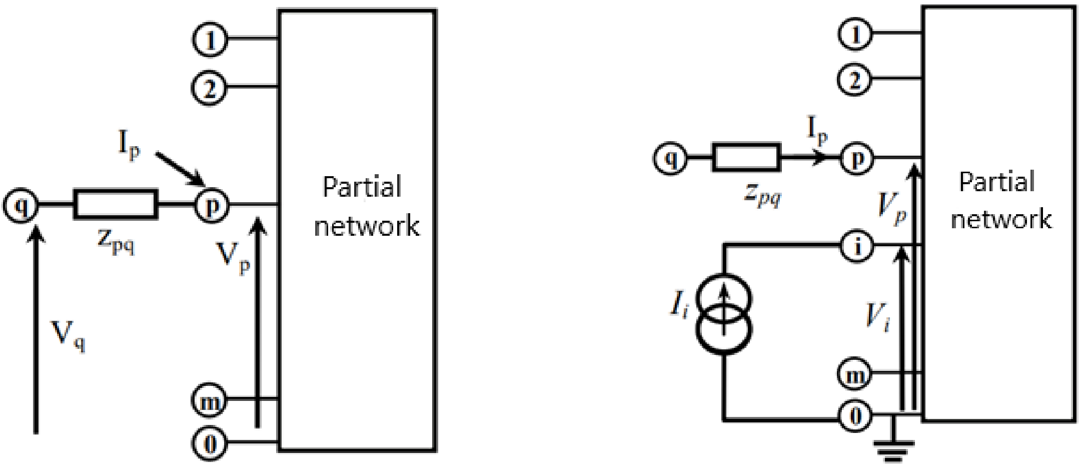


Fig. 11 : Adding a branch

The system of equations can be written in the following form:

$$\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_p \\ \vdots \\ V_m \\ V_q \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1p} & \cdots & Z_{1m} & Z_{1q} \\ Z_{21} & Z_{22} & \cdots & Z_{2p} & \cdots & Z_{2m} & Z_{2q} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\ Z_{p1} & Z_{p2} & \cdots & Z_{pp} & \cdots & Z_{pm} & Z_{pq} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\ Z_{m1} & Z_{m2} & \cdots & Z_{mp} & \cdots & Z_{mm} & Z_{mq} \\ Z_{q1} & Z_{q2} & \cdots & Z_{qp} & \cdots & Z_{qm} & Z_{qq} \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_p \\ \vdots \\ I_m \\ I_q \end{bmatrix} \tag{27}$$

The network elements are passive and linear, therefore:

$$Z_{qi} = Z_{iq} \tag{28}$$

Using the existing matrix Z, elements $Z_{qi} = Z_{iq}$ for $i = 1, \dots, m$ are found by injecting $1pu$ at node i and measuring V_q. If mutual couplings exist between the added element and existing branches, correction terms with admittances $y_{pq,\alpha\beta}$ are used.

Since current was injected only at node i and the currents at the other nodes are zero, equation (27) becomes:

$$\begin{cases} V_1 = Z_{1i} I_i \\ V_2 = Z_{2i} I_i \\ \vdots \quad \vdots \quad \vdots \\ V_p = Z_{pi} I_i \\ \vdots \quad \vdots \quad \vdots \\ V_m = Z_{mi} I_i \\ V_q = Z_{qi} I_i \end{cases} \tag{29}$$

The voltage of the new node can be evaluated based on the voltage V_p and the voltage of the branch V_{pq}.

$$V_{pq} = V_p - V_q \tag{30}$$

or

$$V_q = V_p - V_{pq} \tag{31}$$

The currents in the network elements of Fig. 11 are expressed using the primitive admittances and voltages between these elements.

$$\begin{bmatrix} I_{pq} \\ [I_{\alpha\beta}] \end{bmatrix} = \begin{bmatrix} y_{pq,pq} & [y_{pq,\alpha\beta}] \\ [y_{\alpha\beta,pq}] & [y_{\alpha\beta,\alpha\beta}] \end{bmatrix} \cdot \begin{bmatrix} V_{pq} \\ [V_{\alpha\beta}] \end{bmatrix} \tag{32}$$

pq : fixed index of the element to be added;

$\alpha\beta$: variable index and refers to all elements of the partial network;

$[I_{\alpha\beta}]$ and $[V_{\alpha\beta}]$ are the current and voltage values of the elements of the partial network;

$y_{pq,pq}$: admittance of the added element;

$[y_{pq,\alpha\beta}]$: vector of mutual admittances between the added element $(p - q)$ and the elements $(\alpha - \beta)$ of the partial network;

$[y_{\alpha\beta,\alpha\beta}]$: matrix of primitive admittances of the partial network.

The current in the added branch is zero, therefore:

$$I_{pq} = 0 \tag{33}$$

However, V_{pq} the voltage of this branch is not zero because it is mutually coupled with one or more of the elements of the partial network.

Since :

$$V_{\alpha\beta} = V_\alpha - V_\beta \tag{34}$$

From equations (32), (33) and (34) :

$$I_{pq} = y_{pq,pq} \cdot V_{pq} + [y_{pq,\alpha\beta}] \cdot [V_{\alpha\beta}] = 0 \tag{35}$$

We then obtain

$$V_{pq} = -\frac{[y_{pq,\alpha\beta}] \cdot [V_\alpha - V_\beta]}{y_{pq,pq}} \tag{36}$$

Or :

$$V_q = V_p + \frac{[y_{pq,\alpha\beta}] \cdot [V_\alpha - V_\beta]}{y_{pq,pq}} \tag{37}$$

Furthermore, assuming that the current I_i is equal to unity for V_p , V_q , V_α and V_β of system (29), we finally obtain the term of the new impedance matrix after adding the branch as follows:

$$Z_{qi} = Z_{pi} + \frac{[y_{pq,\alpha\beta}] \cdot [Z_{\alpha i} - Z_{\beta i}]}{y_{pq,pq}} \quad i = 1,2,\dots,m\,;\ i \neq q \tag{38}$$

The element Z_{qq} can be calculated by injecting a current of $1pu$ into node q and measuring the voltage at that same node; the system (29) results in the following:

$$\begin{cases} V_1 = Z_{1q} I_q \\ V_2 = Z_{2q} I_q \\ \vdots \quad \vdots \quad \vdots \\ V_p = Z_{pq} I_q \\ \vdots \quad \vdots \quad \vdots \\ V_m = Z_{mq} I_q \\ V_q = Z_{qq} I_q \end{cases} \tag{39}$$

Let $I_q = 1pu$, so Z_{qq} is known by measuring the voltage V_q. And the current in the added branch will be:

$$I_{pq} = -I_q = -1 \tag{40}$$

From equation (32) :

$$I_{pq} = y_{pq,pq} \cdot V_{pq} + [y_{pq,\alpha\beta}] \cdot [V_{\alpha\beta}] = -1 \tag{41}$$

Hence:

$$V_q = V_p + \frac{1 + [y_{pq,\alpha\beta}] \cdot [V_\alpha - V_\beta]}{y_{pq,pq}} \tag{42}$$

Thus, by replacing V_p , V_q , V_α and V_β from (39) with $I_q = 1pu$, we obtain:

$$Z_{qq} = Z_{pq} + \frac{1 + [y_{pq,\alpha\beta}] \cdot [Z_{\alpha q} - Z_{\beta q}]}{y_{pq,pq}} \tag{43}$$

7.2. Adding a loop

If the added element $p - q$ is a loop, the technique for recalculating the elements of the new impedance matrix involves connecting a dummy voltage source to the added element, as shown in Fig. 12.

However, this technique creates a fictitious node l, which will be subsequently removed. The voltage source e_l is chosen such that the current flowing through the loop will be zero.

The system of equations representing the partial network, the added loop $(p - l)$, and the voltage source $e_l$will be given by:

$$\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_p \\ \vdots \\ V_m \\ e_l \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1p} & \cdots & Z_{1m} & Z_{1l} \\ Z_{21} & Z_{22} & \cdots & Z_{2p} & \cdots & Z_{2m} & Z_{2l} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\ Z_{p1} & Z_{p2} & \cdots & Z_{pp} & \cdots & Z_{pm} & Z_{pl} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\ Z_{m1} & Z_{m2} & \cdots & Z_{mp} & \cdots & Z_{mm} & Z_{ml} \\ Z_{l1} & Z_{l2} & \cdots & Z_{lp} & \cdots & Z_{lm} & Z_{ll} \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_p \\ \vdots \\ I_m \\ I_l \end{bmatrix} \tag{44}$$

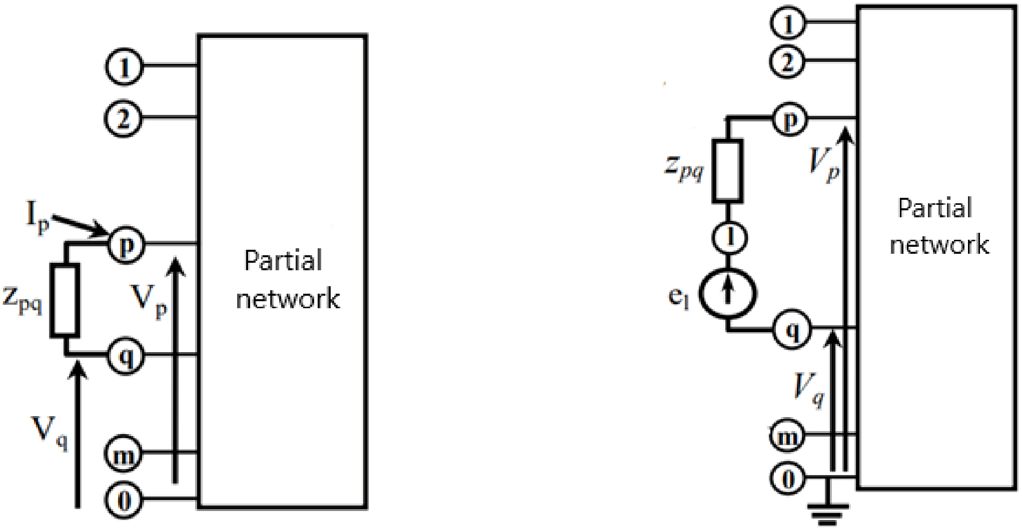


Fig. 12 : Adding a loop

Since the source voltage e_l is equal to (V_l – V_q), then the element Z_{li} can be determined by injecting a current of $1pu$ at node i and measuring the voltages, taking into account node l.

Since all other currents are zero, it follows from equation (44) that:

$$\begin{cases} V_1 = Z_{1i} I_i \\ V_2 = Z_{2i} I_i \\ \vdots \quad \vdots \quad \vdots \\ V_p = Z_{pi} I_i \\ \vdots \quad \vdots \quad \vdots \\ V_m = Z_{mi} I_i \\ e_l = Z_{li} I_i \end{cases} \tag{45}$$

That's to say :

$$V_k = Z_{ki} \cdot I_i \quad i = 1,2, \dots, m \tag{46}$$

$$e_l = Z_{li} \cdot I_i \tag{47}$$

In addition, the voltage source is given by:

$$e_l = V_l - V_q = V_p - V_q - V_{pl} \tag{48}$$

And since the current in the added loop is zero, $I_{pq} = 0$, the element $p - l$ can therefore be treated as a branch. The current in this element will be obtained from (32):

$$I_{pl} = y_{pl,pl}V_{pl} + [y_{pl,\alpha\beta}][\mathrm{V}_{\alpha\beta}] = I_{pq} = 0 \tag{49}$$

Therefore :

$$V_{pl} = -\frac{[y_{pl,\alpha\beta}] \cdot [V_\alpha - V_\beta]}{y_{pl,pl}} \tag{50}$$

And since : $y_{pl,pl} = y_{pq,pq}$ and $[y_{pl,\alpha\beta}] = [y_{pq,\alpha\beta}]$

$$V_{pl} = -\frac{[y_{pq,\alpha\beta}] \cdot [V_\alpha - V_\beta]}{y_{pq,pq}} \tag{51}$$

By setting $I_i = 1pu$ in equation (45) and combining (48) and (51), we obtain:

$$Z_{li} = Z_{pi} - Z_{qi} + \frac{[y_{pq,\alpha\beta}] \cdot [Z_{\alpha i} - Z_{\beta i}]}{y_{pq,pq}} \quad i = 1,2,..,m \ ; \ i \neq q \tag{52}$$

The element Z_{ll} can be calculated by injecting a current at node l with node q as the reference and measuring the voltage at node l with respect to node q.

Since all currents at other nodes are zero, it follows from equation (44) that:

$$V_k = Z_{kl} \cdot I_l \quad k = 1,2,\dots,m \tag{53}$$

$$e_l = Z_{ll} \cdot I_l \tag{54}$$

Assuming $I_l = 1pu$, the current in the element $(p - l)$ is $I_{pl} = -I_l = -1$

$$I_{pl} = y_{pq,pq}V_{pl} + [y_{pq,\alpha\beta}][\mathrm{V}_{\alpha\beta}] = -1 \tag{55}$$

Thus :

$$V_{pl} = \frac{1 + [y_{pq,\alpha\beta}] \cdot [V_\alpha - V_\beta]}{y_{pq,pq}} \tag{56}$$

And after substitution in (48) as before, we find the impedance element Z_{ll} :

$$Z_{ll} = Z_{pl} - Z_{ql} + \frac{1 + [y_{pq,\alpha\beta}] \cdot [Z_{\alpha l} - Z_{\beta l}]}{y_{pq,pq}} \tag{57}$$

The elements of the l-axis (l-row and l-column) of the augmented partial network matrix are given by equations (52) and (57).

It remains to calculate the impedance matrix including the loop effect. This can be accomplished by modifying the elements Z_{ij} $(i,j = 1,2,\dots.m)$ by eliminating the l-axis corresponding to the fictitious node (Kron reduction). This node is eliminated by short-circuiting the voltage source e_l.

From equation (44) :

$$[V] = [Z] \cdot [I] + [Z_{il}] \cdot I_l \qquad i = 1\colon m \tag{58}$$

On the other hand :

$$e_l = \left[Z_{lj}\right] \cdot [I] + Z_{ll} \cdot I_l = 0 \qquad j = 1\colon m \tag{59}$$

From equations (58) and (59), we obtain:

$$[V] = \left([Z] - \frac{\left[Z_{1:m,l}\right] \cdot \left[Z_{l,1:m}\right]}{Z_{ll}}\right) \cdot [I] \tag{60}$$

So it comes:

$$Z^{new} = Z^{old} - \frac{\left[Z_{1:m,l}\right] \cdot \left[Z_{l,1:m}\right]}{Z_{ll}} \tag{61}$$

Or again:

$$Z_{ij}{}^{new} = Z_{ij}{}^{old} - \frac{Z_{il} \cdot Z_{lj}}{Z_{ll}} \tag{62}$$

7.3. Z_{bus} Building Algorithm

Read element data (from/to nodes, primitive and mutual impedances).

For each element k:

- If k is a radial element (branch): apply branch-addition formulas, including mutual terms if present.

$$Z_{qi} = Z_{pi} + \frac{[y_{pq,\alpha\beta}] \cdot [Z_{\alpha i} - Z_{\beta i}]}{y_{pq,pq}}$$

$$Z_{qq} = Z_{pq} + \frac{1 + [y_{pq,\alpha\beta}] \cdot [Z_{\alpha q} - Z_{\beta q}]}{y_{pq,pq}}$$

- If k is a loop element: apply loop-addition formulas, then Kron-reduce the fictitious node.

$$Z_{li} = Z_{pi} - Z_{qi} + \frac{[y_{pq,\alpha\beta}] \cdot [Z_{\alpha i} - Z_{\beta i}]}{y_{pq,pq}}$$

$$Z_{ll} = Z_{pl} - Z_{ql} + \frac{1 + [y_{pq,\alpha\beta}] \cdot [Z_{\alpha l} - Z_{\beta l}]}{y_{pq,pq}}$$

The matrix is then updated by:

$$Z_{ij}{}^{new} = Z_{ij}{}^{old} - \frac{Z_{il} Z_{lj}}{Z_{ll}}$$

Example 5:

Determine the Z_{bus} impedance of the network of Example 4.

The steps for forming the Z_{bus} matrix are performed in the following order: 1-0, 2-0, 1-3, 1-2, 2-3 et 3-4.

1. Element 1-0 : No mutuels.

$$Z = [j1]$$

2. Element 2-0 : branch

$$Z = \begin{bmatrix} j1 & 0 \\ 0 & j0,8 \end{bmatrix}$$

3. Element 1-3 : branch

$Z_{31} = Z_{11} = j1$ $Z_{32} = Z_{12} = 0$ $Z_{33} = Z_{13} + z_{13} = j1 + j0,2 = j1,2$

$$Z = \begin{bmatrix} j1 & 0 & j1 \\ 0 & j0,8 & 0 \\ j1 & 0 & j1,2 \end{bmatrix}$$

4. Element 1-2 : loop

$Z_{l1} = Z_{11} - Z_{21} = j1$ $Z_{l2} = Z_{12} - Z_{22} = 0 - j0,8$ $Z_{l3} = Z_{13} - Z_{23} = j1 - 0$

$$Z_{ll} = Z_{1l} - Z_{2l} + z_{12} = j1 + j0,8 + j0,4$$

$$Z = \begin{bmatrix} j1 & 0 & j1 & j1 \\ 0 & j0,8 & 0 & -j0,8 \\ j1 & 0 & j1,2 & j1 \\ j1 & -j0,8 & j1 & j2,2 \end{bmatrix} \quad \xrightarrow{\text{Elimination axis 4}} \quad Z = \begin{bmatrix} j0,54 & j0,36 & j0,54 \\ j0,36 & j0,509 & j0,36 \\ j0,54 & j0,36 & j0,745 \end{bmatrix}$$

5. Element 2-3 : loop

$Z_{l1} = Z_{21} - Z_{31} = j0,36 - j0.54$ $Z_{l2} = Z_{22} - Z_{32} = j0.509 - j0,36$

$Z_{l3} = Z_{23} - Z_{33} = j0,36 - j0.745$ $Z_{ll} = Z_{2l} - Z_{3l} + z_{23} = j0,149 + j0,385 + j0,2$

$$Z = \begin{bmatrix} j0,54 & j0,36 & j0,54 & -j0,18 \\ j0,36 & j0,509 & j0,36 & j0,149 \\ j0,54 & j0,36 & j0,745 & -j0,385 \\ -j0,18 & j0,149 & -j0,385 & j0,734 \end{bmatrix} \quad \xrightarrow{\text{Elimination axis 4}} \quad Z = \begin{bmatrix} j0,496 & j0,396 & j0,446 \\ j0,396 & j0,479 & j0,438 \\ j0,446 & j0,438 & j0,543 \end{bmatrix}$$

6. Element 3-4 : branch

$Z_{41} = Z_{31} = j0,446$ $Z_{42} = Z_{32} = j0.438$ $Z_{43} = Z_{33} = j0,543$

$Z_{44} = Z_{34} + z_{34} = j0,543 + j0,08 = j0,623$

$$Z = \begin{bmatrix} j0,496 & j0,396 & j0,446 & j0,446 \\ j0,396 & j0,479 & j0,438 & j0,438 \\ j0,446 & j0,438 & j0,543 & j0,543 \\ j0,446 & j0,438 & j0,543 & j0,623 \end{bmatrix}$$